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NƏSİRƏDDİN TUSİNİN

xatirəsinə həsr olunmuş XI Beynəlxalq Konfransın

TEZİSLƏRİ

Modern Problems of Mathematics and Mechanics

ABSTRACTS

of the XI International Conference dedicated to the memory of the genius Azerbaijani scientist and thinker

NASIREDDIN TUSI

July 03-06, 2024, Baku

Institute of Mathematics and Mechanics

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Shamakhy Astrophysical Observatory (Azerbaijan)

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Istanbul Technical University (Turkiye)

A B S T R A C T S

OF THE XI INTERNATIONAL CONFERENCE “MODERN PROBLEMS OF MATHEMATICS AND MECHANICS” DEDICATED TO THE MEMORY OF A GENIUS AZERBAIJANI SCIENTIST AND THINKER NASIRADDIN TUSI

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NASIREDDIN TUSI, OUTSTANDING PERSONALITY, TALENTED TEACHER, ENCYCLOPEDIC SCIENTIST

TUSI'S LIFE

The world-famous scientist Nasireddin Tusi lived and created in the 13th century. He was born on February 17, 1201, according to one legend, in the city of Tus, in the family of Sheikh Wajihaddin Muhammad ibn Hasan. But the famous historian Rashidaddin Fazlullah (1247-1318) writes in his work "Jami al-tawarikh" that his origin is from the city of Hamedan, the former capital of the Seljuk Turks, whose population consists mainly of ethnic Azerbaijanis. Tusi's full name is Muhammad ibn Muhammad ibn Hasan. The name of Nasireddin's is a nickname given to him after acquiring religious knowledge at a high level.

There is very little information about Tusi's childhood and early youth. He spent his childhood and youth in the city of Tus. He learned Quranic sciences, Arabic and Persian languages from his father, and mathematics, logic, and astronomy from well-known scientists of his time in the city of Nishapur.

Tusi married Nargiz in 1230 and three sons were born from this marriage.

His eldest son, Sadreddin Ali, worked at Maragha observatory. He learned mathematics, astronomy and philosophy from his father. After his father's death, he headed the Maraga observatory, and in general, after Tusi, all the responsibilities fell on him.

His second child, Asladdin, was also known for his high education. He was also involved in politics. Being a good engineer, he closely participated in the construction of the observatory. Rashidaddin Fazlullah writes that Ghazan Khan and Olcaytu Khan believed in Asladdin very much. After the death of his elder brother Sadreddin, Olcaytu entrusted the management of Maragha observatory to Asladdin.

Abulgasim, the third son, was born in Maraga. Like his brothers, he was a talented scientist who mastered many sciences.

Tusi's children directly participated in the recording and transfer of his scientific heritage in manuscript form. At present, there are different copies of many of Tusi's works in the libraries of different countries of the world.

More than 25 years of his life were spent in Kuhistan, the state of the Ismailis, including more than 20 years in the Alamut fortress.

It is clear from Tusi's writing at the end of his work "Sharh-al-Isharat" that he lived in the Alamut fortress for more than 20 years in difficult conditions. Nevertheless, he wrote commentaries on the works of Euclid, Ptolemy, Archimedes, "Akhlaqi-Nasiri", "Sharh-al-Isharat", "Tahriri-Euglidis" and others. He wrote such works just in this period.

In 1256, Hulaku Khan, the grandson of Genghis Khan, took Tusi as his chief advisor after taking the Alamut fortress. Hulaku Khan consulted with him on almost all issues of state importance. It is said that before marching on Baghdad, Hülaku took advice from Tusi. All Hulaku Khan's advisors, led by the vizier Salar, wanted to dissuade him from attacking Baghdad. They said that if he attacked the Abbasids and killed the caliph, many disasters would occur on earth, and the worst thing was that Hulaku Khan himself might die that very year. When Hulaku Khan wanted to know Tusi's attitude to this issue, he said: "When the caliphs from the Abbasid dynasty: Muhammad Amin, Mutawakkil, Muntasir were killed, no disaster happened, and now if you take Baghdad and kill Caliph Mustasim, nothing will happen." Heeding the advice of Tusi, Hulaku Khan he attacked Baghdad, took it for a short time, and thus ended the rule of the Abbasids, which had ruled for 500 years.

Some eastern historians condemn this action of Tusi, accusing him of betraying the religion of Islam. But this idea is wrong. Nasireddin Tusi is a genius scholar who highly appreciates Islamic values, propagates them, and has written many works on the Islamic religion.

Some people associate Tusi's action with his sect affiliation: Tusi was a Shia, and the sect led by the caliph and caliphate was Sunni. But these ideas are also wrong and have no basis. Even though Tusi tried to spread Shiism and developed the Imami sect, he maintained a friendly relationship with all Islamic sects and their representatives, he never judged them because they were Sunni or Shia, he tried to solve the problems faced by all Muslims as much as possible, he always defended the right and the truth.

Nasireddin Tusi as a personality

Besides being an encyclopedic scientist, Nasireddin Tusi was a person with very high qualities. One of the contemporaries of that time writes about his humanity and beautiful features: "If Nasireddin Tusi impressed scientists with his knowledge and erudition, he won the respect and love of ordinary people with his social qualities - justice, culture, and good manners. He treated everyone who approached him as a friend. His main characteristics were kindness, humility, kindness, gentle, but at the same time restrained and patient behavior, which made even his staunchest enemies respect him. He is the gentle father in whom we rely. Sometimes we want to annoy him so that we can see his angry state, but we still feel pity for him. To serve the motherland, we entrust our children and our homes to him. If someone finds him, he loses nothing, and whoever loses him loses everything."

The prayer he wrote at the beginning of the work "Esasul-Iktibas" is one of the examples that characterize Tusi's personality: "God, increase my knowledge." Help those who learn divine wisdom (devotees of knowledge) to understand the truth, to understand the righteousness and to achieve good. Spend their earnest effort on attaining perfection, seeking merit, and surrounding themselves with virtue, so that they may be firm in truth, free from crookedness, reliable in certainty, weary of doubt, kneaded with knowledge, afraid of ignorance, confessing their faults, and avoiding arrogance. [Them] Predominance of one's own opinion, partiality, pride, flattery, vanity, oppression, disobedience, corruption, foolishness, enmity, sedition, inclining to falsehood, arrogance, cunning, misleading, denying the truth and turning away from it, insisting on falsehood and not seeing it. Keep away from the characteristics of learning science, quarreling, telling lies, misleading and ruining for the sake of coming, superiority and pride. May they stay away from the deceitful trick of imitation, deceptive doubt, vain desire, and following a path that does not agree with you. Give them a chance so that they recognize the rights of past and present virtues and envy does not hinder them on this path. They should teach them the gratitude of the gift of wisdom in accordance with the talents of others, and in this way they should not allow errors such as envy, competition and negligence. Also, they should stay away from laziness, unemployment, wasting time and endangering their lives, and be firm by cooperating with perfect religion and the right path. So, their ultimate goal should be to attain eternal glory and reach God."

From the requests in this prayer, it is clear how much Tusi values science, scientists, and education, and that people engaged in science should stay away from crookedness, arrogance, ignorance, pride, and flattery. Tusi asks the Great God not for sustenance, health, wealth, but for increasing his knowledge.

Tusi's high appreciation of science, education, free and proud living in life is also reflected in his poems.

Nasireddin Tusi as a teacher

Nasireddin Tusi is also known as a skilled teacher. He had many students in various fields of science. Let's give brief information about some of them.

1. Allama Hilli (Jamaladdin nicknamed Abu Mansur Hasan Ibn Yusif Ibn Mutahhar al-Hilli) was born in 1250 in the city of Hilla, east of the Euphrates River, between the cities of Kufa and Najaf. He was the first person to receive the title of Ayatollah. Allama Hilli is the author of more than 120 works in various fields of science. He wrote commentaries on the works of a number of scientists, including his teacher Nasreddin Tusi, such as "Tajrid al-mantig" and "Tajrid al-etigad". Allama Hilli died in 1324 in the city of Hilla.

2. Qutbuddin Shirazi (Abu Sana Mahmud Ibn Masud Ibn Muslihiddin Kazeruni Shirazi) was born in 1235. He is considered one of Tusi's most talented students. Shirazi, who studied medicine from his father from a young age, lost his father when he was 14 years old and started working as a doctor instead of his father at "Muzaffari" hospital in Shiraz. Then he visited different cities and studied logic, wisdom, and medical sciences from well-known scientists of the time. Shirazi, who came to Maragha in 1260 at the invitation of Tusi, studied mathematics and astronomy with him. Tusi called him "gutbul-falakul-vujud" that is "the pole of the sky of existence".

3. Kamal al-Din Baghdadi (Kamal al-Din Abdurrazzag ibn Ahmad Sheybani Baghdadi) was born in 1244 in Baghdad. As a young man, he worked in government affairs in Baghdad during the reign of the last Abbasid caliph. When Hulaki Khan took Baghdad, he was taken prisoner, a few years later he was released from captivity with the help of Tusi and worked with him for more than ten years. He was in charge of the library of Maragha observatory for a long time and returned to Baghdad only after Tusi's death. Kamal al-Din Baghdadi was one of the famous historians of the thirteenth century. The fifty-volume work "Majmaul-adab fi mojamul-asma wa-l-algab" ("Collection of etiquette rules in the dictionary of names and nicknames") is one of his important works.

4. Astrabadi (Abu Fazail Hasan ibn Muhammad ibn Sharafshah Alavi Astarabadi) was Tusi's most beloved and capable student and colleague after Qutbuddin Shirazi. He took lessons from Tusi in Maragha city. When Tusi went to Baghdad in 1274, Astarabadi accompanied him, and after Tusi's death, he taught at the Nuriya school in Mosul and wrote commentaries on a number of his master's works. Astrabadi died on May 20, 1315 and was buried in Tabriz.

5. Meysam Bohrani (Kamaladdin Meysam Ibn Ali Ibn Meysam Bohrani) was a mathematician, doctor and jurist. He wrote a commentary on "Nahjul Balagha" written by Imam Ali.

There were other students of Tusi: Ibrahim Hamavi Cuyini. Asireddin Omani, Majideddin Tusi, Majideddin Maraghi.

One of the factors showing Tusi's recognition as a talented teacher is his knowledge of mathematics, logic, astronomy, ethics, etc. that most of the works written in the field are in textbook style. Among these books, "Commentary of Euclid" on geometry, "A collection of calculations with the help of a dusty board" on arithmetic, "Treatise on complete quadrilaterals" on trigonometry, "Fundamentals of learning science" on logic have been used as textbooks in educational institutions of different countries for a long time.

Tusi's scientific creativity

Tusi is the author of more than 150 works related to almost all fields of science. Of course, it is impossible to list and evaluate the contributions of a genius scientist to science in these works one by one. A detailed, objective analysis of Tusi's work is a subject that is wide and complex enough to be the subject of one or more monographs. I should mention that two monographs written about Tusi's works in mathematics and logic were presented to the conference participants.

Historians of science of the world showed great interest and highly appreciated Tusi and his work at different times. In Azerbaijan, Turkey, Russia, Uzbekistan, the Islamic Republic of Iran and other countries, many works have been written dedicated to the life and work of the encyclopedic scientist (Mudarris Rezavi, Mammadbeyli, Sayali, Dilgan, Rosenfeld, Yushkevich, Matviyevskaya and others). Starting from the 16th century, Tusi's works began to spread in Europe. In 1594, the publication of the great thinker's work

"Euclid's Commentary" in Rome gave a strong impetus to the development of the science of geometry in Europe. Since Islam was widespread in Spain in the Middle Ages, most European scholars knew Arabic. Therefore, despite the work being in Arabic, European mathematicians could benefit from this work. The publication of the work in Latin in London in 1657 made it possible for almost all European scholars to use this work.

In addition to "Euclid's Commentary", the genius scientist's "Treatise on Complete Quadrilaterals", "Removal of Doubts on Parallel Lines", "Tazkira", "A collection of calculations with the help of a dusty tablet", "Twenty Chapters on Astrolabia", "Zici-Elkhani", " On the reflection and refraction of light", "Basics of mastering the sciences", "Isolation al-mantig" and others. His works brought many innovations to arithmetic, trigonometry, geometry, physics, logic and played an important role in the development of these sciences in Europe.

Some European scientists claim that the scientists of the medieval Islamic world, including Tusi, did not bring any innovations to science, they only translated the works of ancient Greek scientists into Arabic and played a role in their spread in Europe. But this idea, r is completely wrong. Suffice it to say that the research carried out in our institute in recent years shows that in the field of mathematics, the definition of a real positive number 300 years before Newton, the axioms of stereometry 400 years before the Italian mathematician Giovanni Borelli, to denote power by symbols 350 years before the English mathematician William Outred, whether the rule of "criteria" is necessary or sufficient to verify the correct performance of arithmetic operations, 300 years before the Italian Nicola Tartali and the German Christopher Clavius, taking the least common divisor of the denominators for the common denominator when operations on fractions are performed, 200 years before Nicola Schucke and 300 year before Luca Pacoli was brougth to science by N.Tusi. Until the 16th century, mathematicians did not count units as numbers. Tusi gave a different definition of integer than Euclid, and according to this definition, unit is also considered a number. In addition, signs of Tusi's creativity are evident in the works of European scientists of the 13th-17th centuries.

When speaking about Nasireddin Tusi, one cannot fail to mention the Maraga observatory created by him.

The success of the Baghdad campaign further increased Tusi's influence with Hulaku Khan. Using this, Tusi began to realize his long-standing dream of creating a large scientific center, an observatory in Azerbaijan. In 1259, the foundation of Maraga observatory was laid. It should be noted that this observatory, which has a rich library and modern astronomical devices for its time, has been the most famous science center in the world for a long time. Famous scientists from all over the world were invited to work here. Suffice it to say that after the Maraga observatory ceased its activity, the scientists working here created and managed observatories in different parts of the world. Tabriz, Beijing, Damascus, Samarkand observatories can be cited as examples.

Tusi headed the Maraga observatory for 15 years, where he trained many scientists and wrote important works that stimulated the development of astronomy. According to the opinions of well-known acronomists (Arthur Berry, Mikhail Fyodorovich Subbotin), the astronomical devices developed on the basis of Tusi's guidance and drawings are new and very perfect. Arthur Berry writes in his book "A Brief History of Astronomy" that the instruments Copernicus made and used were much inferior to those of Tusi.

Tusi's work has been studied by historians of science in many countries since his time. Among his students, Allama Hilli, Qutbuddin Shirazi, Nizamuddin Nishapuri, Kamaluddin Farsi, Najmuddin al-Katibi al-Qazvini, Seyyid Ruknuddin, Kamaluddin Baghdadi and others studied his works and wrote extensive commentaries on Tusi's works even during his lifetime.

European and American scientists Henrich Zutter (1848-1922), Edward Brown (1862-1926), Karl Brockelman (1868-1956), Eilhard Wiedeman (1852-1928), Jean Baptiste DeLambert (1749-1822), Ignati Yulianovich Krachkovsky (1883) -1951), Aldo Mieli (1879-1950), Georg Sarton (1884-1956), Charles Ambrose Storey (1888-1967), Henry Georg Farmer (1882-1965), Stormman, Julius Rushka (1867-1949) and other well-known Historians of science commented on the works of Nasireddin Tusi in mathematics, astronomy, logic, philosophy and other fields of science and wrote articles about his life.

Among the Turkish scientists, Ghazizade Rumi (1364-1436), a teacher of Ulugbey, one of the heads of the Samarkand observatory, his grandson Mirim Chalabi (1450-1525), the famous historian and geographer Mustafa Haji Khalifa (1609-1657), Muhammad al-Bursali (1861-1926), Hamit Dilgan, Qardi Hafiz Tukan (1910-1971), AydIn Sayili (1913-1993) and others studied the works of Nasireddin Tusi and translated some of them into Turkish.

In Azerbaijan, starting from the middle of the 20th century, Tusi's works in the fields of mathematics, astronomy, logic, and economics are being studied more.

In the session dedicated to the 750th anniversary of Tusi's birth in Baku on November 16-22, 1951, Habibulla Mammadbeyli, Zahid Khalilov, Boris Rosenfeld, Ashraf Huseynov, Rustam Sultanov and others gave a report on the works of the great thinker in the fields of mathematics, astronomy, economics, logic, and philosophy.

In 2001, under the organization of ANAS, the 800th anniversary of the great thinker of Azerbaijan was celebrated with great solemnity. An International Conference dedicated to this jubilee was held in the observatory named after Tusi in Pirgulu.

In addition, in 2011 and 2014, more than 20 countries (USA, Russia, Turkey, Iran, China, Canada, England, Spain, France, Switzerland, Ukraine, Algeria, Indonesia, Jordan, Uzbekistan, Tajikistan, Georgia, etc.)) international conferences dedicated to the 810th anniversary of Nasireddin Tusi's birth and 740th anniversary of his death were held in Baku with the participation of leading scientists in the history of science.

In recent years at the Institute of Mathematics and Mechanics, the works written by Tusi in the field of mathematics, logic, "A collection of calculations with the help of a dusty board", "Calculation, algebra and countermeasures", "Commentary on the book "Conjectures" by Sabit ibn Qurra", "Commentary on Euclid's work "Data". "Isolation al logic", "Isolation al etigad", "Esasul igtibas", etc. were translated from Arabic and Persian into Azerbaijani, and some into Russian.

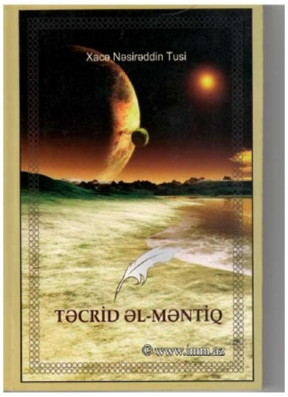
Main services of Nasireddin Tusi

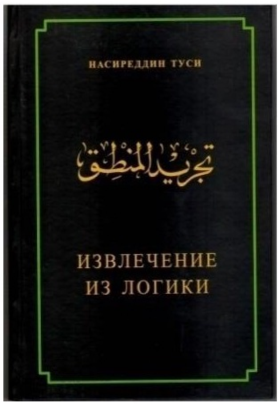
1. Writing works that promote the development of all fields of science, especially mathematics, astronomy, and logic.
2. Bringing many innovations to science in these works.
3. Building and equipping the Maraga Observatory.
4. Creation of a large library at Maragha Observatory.
5. Raising talented students in all fields of science.

Nasiruddin Tusi died on June 25, 1274, and was buried near the grave of the seventh imam Museyi Kazim.

These words are written on his grave:

The king of the world of science, the sultan of scholars, the support of the right, the people, and the religion, Muhammad ibn Muhammad Tusi.





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ALGEBRA, GEOMETRY AND TOPOLOGY

Introduction to neutrosophic topology on soft sets

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In this work, we give the definition of neutrosophic topology (cotopology), which is a mapping satisfying some definite conditions from $SS(X,E)$ to $[0,1]$. We show that a neutrosophic topological space gives a parameterized family of soft tritopologies on X. Then we introduce the concepts of base and subbase of neutrosophic topological spaces on soft sets.

Definition 1. A mapping $\tau = (\tau_T, \tau_I, \tau_F) : SS(X,E) \to [0,1]$ is called a neutrosophic topology on X if the following conditions hold:

1. For $\forall (F,E) \in SS(X,E)$, $\tau_T(F,E) + \tau_I(F,E) + \tau_F(F,E) \leq 3$;
2. $\tau_T(\Phi) = \tau_T\left(\tilde{X}\right) = 1,\ \tau_I(\Phi) = \tau_I\left(\tilde{X}\right) = 1,\ \ \tau_F(\Phi) = \tau_F\left(\tilde{X}\right) = 0$
3. For $\forall (F,E), (G,E) \in SS(X,E)$,
$$\begin{aligned}\tau_T\left((F,E)\tilde{\bigcap}(G,E)\right) &\geq \tau_T(F,E) \wedge \tau_T(G,E),\\ \tau_I\left((F,E)\tilde{\bigcap}(G,E)\right) &\geq \tau_I(F,E) \wedge \tau_I(G,E),\\ \tau_F\left((F,E)\tilde{\bigcap}(G,E)\right) &\leq \tau_F(F,E) \vee \tau_F(G,E).\end{aligned}$$
4. For $\forall (F_i,E) \in SS(X,E),\ i \in \Delta$
$$\tau_T\left(\bigcup_{i\in\Delta}(F_i,E)\right) \geq \bigwedge_{i\in\Delta}\tau_T(F_i,E),\ \ \tau_I\left(\bigcup_{i\in\Delta}(F_i,E)\right) \geq \bigwedge_{i\in\Delta}\tau_I(F_i,E),$$
$$\tau_F\left(\bigcup_{i\in\Delta}(F_i,E)\right) \leq \bigvee_{i\in\Delta}\tau_F(F_i,E),$$

The triple (X, E, τ) is called a neutrosophic topological space of soft sets. Neutrosophic topological space (X, E, τ) is denoted by NTS.

Theorem 1. *a) If $\tau = (\tau_T, \tau_I, \tau_F)$ is a neutrosophic topology on X, then $\upsilon = (\upsilon_T, \upsilon_I, \upsilon_F)$ is a NCT on X such that $\upsilon_T(F,E) = \tau_T\left((F,E)^C\right)$, $\upsilon_I(F,E) = \tau_I\left((F,E)^C\right)$, $\upsilon_F(F,E) = \tau_F\left((F,E)^C\right)$.*

b) If $\upsilon = (\upsilon_T, \upsilon_I, \upsilon_F)$ is a NCT on X, then $\tau = (\tau_T, \tau_I, \tau_F)$ is a neutrosophic topology on X such that $\tau_T(F,E) = \upsilon_T\left((F,E)^C\right)$, $\tau_I(F,E) = \upsilon_I\left((F,E)^C\right)$, $\tau_F(F,E) = \upsilon_F\left((F,E)^C\right)$.

Theorem 2. *Let (X, E, τ) be a NTS. For each $r \in (0, 1]$,*

$$\begin{aligned}
&\tau_{T_r} : \{(F,E) \in SS(X,E) : \ \tau_T(F,E) \geq r\},\\
&\tau_{I_r} : \{(F,E) \in SS(X,E) : \ \tau_I(F,E) \geq r\},\\
&\tau_{F_r} : \{(F,E) \in SS(X,E) : \ \tau_F(F,E) \geq 1-r\},
\end{aligned}$$

are three descending families of soft topologies of soft sets on$X$.

Theorem 3. *Let $\{(\sigma_{T_r}, \sigma_{I_r}, \sigma_{F_r})\}_{r\in(0,1]}$ be a descending family of soft tri-topologies on X. Then $\tau_T(F,E) = \vee\{r : (F,E) \in \sigma_{T_r}\}$, $\tau_I(F,E) =$ $= \vee\{r : (F,E) \in \sigma_{I_r}\}$,$\tau_F(F,E) = \wedge\{1-r : (F,E) \in \sigma_{F_r}\}$, are a $NFT's$.*

Definition 2. Let (X, E, τ) be a NTS.

$(\beta_T, \beta_{\mathrm{I}}, \beta_F) : SS(X,E) \to [0;1]$ is a called a base of (τ_T, τ_I, τ_F) if the following conditions hold: $\forall (F,E) \in SS(X,E)$,

$$\tau_T(F,E) = \bigvee_{\bigcup\limits_{i\in\Delta}(G_i,E)=(F,E)} \bigwedge_{i\in\Delta} \beta_T(G_i,E),$$

$$\tau_{\mathrm{I}}(F,E) = \bigvee_{\bigcup\limits_{i\in\Delta}(G_i,E)=(F,E)} \bigwedge_{i\in\Delta} \beta_{\mathrm{I}}(G_i,E),$$

$$\tau_F(F,E) = \bigwedge_{\bigcup\limits_{i\in\Delta}(G_i,E)=(F,E)} \bigvee_{i\in\Delta} \beta_F(G_i,E).$$

Theorem 4. *Let (X, E, τ) be a NTS and $Y \subset X$. Define two mappings $(\tau_{T_Y}, \tau_{I_Y}, \tau_{F_Y}) : SS(Y,E) \to [0,1]$ by:*

$$\begin{aligned}
\tau_{T_Y}(F,E) &= \vee\left\{\tau_T(G,E) : (F,E) = (G,E)\tilde{\bigcap}\tilde{Y}, (G,E) \in SS(X,E)\right\},\\
\tau_{I_Y}(F,E) &= \vee\left\{\tau_I(G,E) : (F,E) = (G,E)\tilde{\bigcap}\tilde{Y}, (G,E) \in SS(X,E)\right\},\\
\tau_{F_Y}(F,E) &= \wedge\left\{\tau_F(G,E) : (F,E) = (G,E)\tilde{\bigcap}\tilde{Y}, (G,E) \in SS(X,E)\right\}.
\end{aligned}$$

Then the triplet $(\tau_{T_Y}, \tau_{I_Y}, \tau_{F_Y})$ *is a neutrosophic topology on* Y *and*

$$\tau_{T_Y}\left((G,E)\tilde{\bigcap}\tilde{Y}\right) \geq \tau_T(G,E), \tau_{I_Y}\left((G,E)\tilde{\bigcap}\tilde{Y}\right) \geq \tau_I(G,E),$$

$$\tau_{F_Y}\left((G,E)\bigcap\tilde{Y}\right) \leq \tau_F(G,E).$$

Theorem 5. *Let* $\{(X_\lambda, E_\lambda, \tau_\lambda)\}_{\lambda\in\Lambda}$ *be a family of NTSs, different* X'_λ *be disjoint. Then* (τ_T, τ_I, τ_F) *which is defined by:*

$$\tau_T(F,E) = \underset{\lambda\in\Lambda}{\wedge} \tau_{T_\lambda}\left((F,E)_\lambda\right),$$

$$\tau_I(F,E) = \underset{\lambda\in\Lambda}{\wedge} \tau_{I_\lambda}\left((F,E)_\lambda\right),$$

$$\tau_F(F,E) = \underset{\lambda\in\Lambda}{\vee} \tau_{F_\lambda}\left((F,E)_\lambda\right), \forall (F,E) \in \left(\tilde{X}, E\right)$$

is a neutrosophic topology on X*.*

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On polyform Abelian groups

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Let R be an associative ring with identity. A submodule N of an R-module M is said to be *rational* if $\mathrm{Hom}(M/N, E(M)) = 0$, where $E(M)$ is an injective envelope of M. A module M is *polyform* if every essential subgroup of M is rational (see [1] and [2]). Clearly, all semisimple and nonsingular modules are polyform. We will show that there are no other polyform modules if $R = \mathbb{Z}$.

Theorem *An abelian group A is polyform if and only if A is semisimple or torsıon-free.*

A pure submodule N of a module M that is rational is called a P-*rationa*l submodule ([3]). A module M is P-*polyform* if every essential submodule is P-rational ([4]). We will also give some properties of P-rational subgroups and P-polyform abelian groups.

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Intersections of an ellipsoid and its caustics

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In the talk, we will discuss the results from [1] about the parametrization and the classification of all possible cases of the intersections of a triaxial ellipsoid with its caustics introduced by Cayley [2]. It is proved that there are in total 8 different cases of these intersections. One of the applications of these results is the solution of Apollonius' problem on the number of normals for the points of the ellipsoid [3,4]. Also related problem about the position of the ellipsoid with respect to its caustics is studied. All possible 12 cases are demonstrated:

(i) $a^2 < 2c^2$, $a^2 + c^2 < 2b^2$;
(ii) $a^2 < 2c^2$, $a^2 + c^2 > 2b^2$;
(iii) $b^2 < 2c^2 < a^2$, $a^2 + c^2 < 2b^2$;
(iv) $b^2 < 2c^2 < a^2$, $a^2 + c^2 > 2b^2$, $a^2 < 2b^2$;
(v) $b^2 < 2c^2 < a^2$, $a^2 > 2b^2$;
(vi) $b^2 > 2c^2$, $a^2 + c^2 < 2b^2$, $\frac{1}{a^2} + \frac{1}{c^2} < \frac{3}{b^2}$;
(vii) $b^2 > 2c^2$, $a^2 + c^2 > 2b^2$, $a^2 < 2b^2$, $\frac{1}{a^2} + \frac{1}{c^2} < \frac{3}{b^2}$;
(viii) $b^2 > 2c^2$, $a^2 > 2b^2$, $\frac{1}{a^2} + \frac{1}{c^2} < \frac{3}{b^2}$;
(ix) $b^2 > 2c^2$, $2b^4 + 2c^4 - a^2b^2 - a^2c^2 - 2b^2c^2 > 0$;
(x) $b^2 > 2c^2$, $a^2 + c^2 < 2b^2$, $\frac{1}{a^2} + \frac{1}{c^2} > \frac{3}{b^2}$, $2b^4 + 2c^4 - a^2b^2 - a^2c^2 - 2b^2c^2 < 0$;
(xi) $b^2 > 2c^2$, $a^2 + c^2 > 2b^2$, $a^2 < 2b^2$, $\frac{1}{a^2} + \frac{1}{c^2} > \frac{3}{b^2}$;
(xii) $b^2 > 2c^2$, $a^2 > 2b^2$, $\frac{1}{a^2} + \frac{1}{c^2} > \frac{3}{b^2}$.

The following theorems are proved.

Theorem 1. *For ellipsoid* $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, $(c \le b \le a)$ *and its caustics, the following cases are possible:*

1. *If* $a^2 < 2c^2$ *then there are no intersections of the caustics with the ellipsoid,*
2. *If* $b^2 < 2c^2 \le a^2$ *then only one of the caustics intersects the ellipsoid,*
3. *If* $b^2 \ge 2c^2$, *then both of the caustics intersects the ellipsoid.*

In all cases, for the points of the ellipsoid lying outside of the two caustics $n(A) = 2$*, for the points of the ellipsoid lying in only one of these caustics* $n(A) = 4$*, for the points of the ellipsoid lying in both of these caustics* $n(A) = 6$*, and for the intersection points of the ellipsoid and these caustics* $n(A) = 3$ *or* 5*, except some of the points of the ellipsoid, where the caustics intersect each other or these caustics intersect the coordinate planes.*

Theorem 2. *The intersection of ellipsoid* $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ $(c \leq b \leq a)$*, and its caustics is parametrized through*

$$(x(t))^2 = \frac{a^2(a^2+t)^3((b^2+c^2)t+3b^2c^2)}{(a^2-c^2)(a^2-b^2)((a^2b^2+a^2c^2+b^2c^2)t+3a^2b^2c^2)},$$

$$(y(t))^2 = \frac{b^2(b^2+t)^3((a^2+c^2)t+3a^2c^2)}{(b^2-a^2)(b^2-c^2)((a^2b^2+a^2c^2+b^2c^2)t+3a^2b^2c^2)},$$

$$(z(t))^2 = \frac{c^2(c^2+t)^3((a^2+b^2)t+3a^2b^2)}{(c^2-a^2)(c^2-b^2)((a^2b^2+a^2c^2+b^2c^2)t+3a^2b^2c^2)},$$

where $-a^2 \leq t \leq -b^2$ *and* $-b^2 \leq t \leq -c^2$ *correspond to the two intersections of the caustics with the ellipsoid.*

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Modules that generates their simple subfactors

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A ring R is said to be a right H-ring if the injective hulls of nonisomorphic simple right R-modules are homologically independent, that is, for each nonisomorphic simple right R-modules S_1 and S_2, $Hom(E(S_1), E(S_2)) = 0$, where $E(S_1)$ and $E(S_2)$ are the injective hulls of S_1 and S_2, respectively (see, [5]). Right V-ring, commutative Noetherian rings, and commutative semiartinian rings are examples of H-rings (see, [2]). Right Artinian rings that are right H-ring are characterized in [4].

We observe the following characterization of right H-rings: R is a right H-ring if and only if every simple subfactor of $E(S)$ is isomorphic to S for each simple right R-module S. By a subfactor N of a right R-module M, as usual, we mean a submodule of a factor module of M i.e. $N = \frac{A}{B} \subseteq \frac{M}{B}$.

Motivated by the aforementioned characterization of right H-rings, we call a right R-module M right SFI-module if every simple subfactor of M is isomorphic a simple factor of M. In this talk, we shall present the following results about SFI-modules.

We begin with the characterization of the rings all of whose right modules are SFI. Recall that, a ring R is said to be right max if every nonzero right R-module has a maximal submodule.

Theorem 1. *The following statements are equivalent for a right R :*

(1) Every right R-module is an SFI-module.

(2) R is a right H-ring and right max-ring.

Theorem 2. *The following statements are equivalent for a ring R.*

(1) Right SFI-modules are closed under factor modules.

(2) Right SFI-modules are closed under submodules.

(3) Right SFI-modules are closed under direct summands.

(4) R is right H-ring and right max.

A ring R is right Kasch if every simple right R-module embeds in R. Dually, a ring R is said to be right dual Kasch if every simple right R-module is a homomorphic image of an injective right R-module, see [1]. A commutative ring is said to be a classical ring if every element is either a zero-divisor or a unit.

It is clear that R is a right and left SFI-module. The SFI assumption on $E(R)$ gives the following over commutative noetherian rings.

Theorem 3. *Let R be a commutative Noetherian ring. The following statements are equivalent.*

(1) $E(R)$ is an SFI-module.

(2) R is a classical ring.

(3) R is a dual Kasch ring.

(4) R is a Kasch ring.

Theorem 4. *The following statements are equivalent for a right Artinian ring R.*

(1) Every right R-module is an SFI-module.

(2) Every finitely generated right R-module is an SFI-module.

(3) Every cyclic right R-module is an SFI-module.

(4) $Ext_R(S_1, S_2) = 0$ for each nonisomorphic simple right R-modules S_1, S_2.

(5) R is a right H-ring.

Now, we give a characterization of SFI-modules over the ring of integers. For an abelian group M, as usual, $T(M)$ and $T_p(M)$ are the torsion subgroup and the p-primary components of M, respectively. The abelian groups that are SFI are characterized as follows in terms of the torsion submodule and the p-primary components.

Theorem 5. *The following are hold for an abelian group M.*

(1) A torsion abelian group M is an SFI-module if and only if $pM \neq M$ for each prime p with $T_p(M) \neq 0$.

(2) If $T(M) \neq M$, then M is an SFI-module if and only if $pM \neq M$ for each prime p.

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On some topological indices for the orbit graph of Dihedral group

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Let Γ_G be the orbit graph of G, with non-central orbits in the subset of order two commuting elements in G, and the vertices of Γ_G connected if they are conjugate. The main objective of this study is to compute several topological indices for the orbit graph of a dihedral group, including the Wiener index, the Zagreb index, the Schultz index, and others. We also find a relationship between the Wiener index and the other indices for the orbit graph. Furthermore, their polynomials have also been computed.

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On Rings determined by injectivity domains of modules

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Injective modules have the class of all modules as their domain of injectivity and are considered to be wealthy in this manner. As an opposite of injectivity, poor modules are introduced in [1] as modules having semisimple modules as their domain of injectivity. The first wave in this line of research has focused mainly on the rings with no middle class (over which every right module is either poor or injective).

In [4], it is shown that the existence of semisimple poor modules is related to the coincidence of the class of all crumbling modules (exactly locally Noetherian V-modules) and the class of all semisimple modules. Using crumbling modules, rings that have middle classes are considered in [3]. Examples of rings that can have more than one middle class is given in the same work.

In this talk, we consider rings that have one middle class of injectivity domains. Especially, we focus on the rings that have the class of crumbling modules as their only middle class of injectivity domains (which we call CMC-ring) (see [2]). Here is a result about such rings.

Theorem 1. **[2, Theorem 7]** *Let R be a right CMC-ring. Then R is either a right V-ring or right Noetherian.*

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Probabilistic four-valued classical logic C4

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Based on the four-valued classical logic C4 a probabilistic four-valued logic has been created.

Many-valued [1] logics are basically construed as truth-functional calculi: the degree of truth of a formula can be calculated from the degrees of truth of its constituents. It has been tempting for many authors to exploit many-valued logic calculi for belief propagation. However, even if this approach may properly work in some expert systems under severe restrictions of the structure of the knowledge base, the notion of compositional logic of uncertainty leads to mathematical difficulties in general. Namely, in a Boolean setting, the uncertainty of a formula cannot be a function of the uncertainties of its constituents.

In this article, we will build a probabilistic logic based on the four-valued classical logic C4 in a different way. We will define the probability of statements based on the truth table and the algorithm for calculating the probabilities of any formula.

Main Content. In four-valued classical logic C4 [2], each statement takes 4 values: 0, 1, 2, 3.

We assume that each statement (x) takes values 0, 1, 2, 3 with corresponding probabilities: $(P_x(0), P_x(1), P_x(2), P_x(3))$ and

$$P_x(0) + P_x(1) + P_x(2) + P_x(3) = 1$$

Table 1: Truth Table $\neg x$

x	$\neg x$
0	3
1	2
2	1
3	0

Then the probability:

$$P_{\neg x}(0) = P_x(\neg 0) = P_x(3)$$

$$P_{\neg x}(1) = P_x(\neg 1) = P_x(2)$$

$$P_{\neg x}(2) = P_x(\neg 2) = P_x(1)$$

$$P_{\neg x}(3) = P_x(\neg 3) = P_x(0)$$

Table 2: Truth Table $x \wedge y$

$\wedge$	0	1	2	3
0	0	0	0	0
1	0	1	0	1
2	0	0	2	2
3	0	1	2	3

Similarly, probabilities for other logical operations can be defined. For instance, the probability for $x \wedge y$ is given by:

$$P_{x\wedge y}(0) = \sum_{i=0}^{3}\sum_{j=0}^{3}\{P_x(i) \times P_y(j) \mid i \wedge j = 0\}$$

$$P_{x\wedge y}(1) = \sum_{i=0}^{3}\sum_{j=0}^{3}\{P_x(i) \times P_y(j) \mid i \wedge j = 1\}$$

$$P_{x\wedge y}(2) = \sum_{i=0}^{3}\sum_{j=0}^{3}\{P_x(i) \times P_y(j) \mid i \wedge j = 2\}$$

$$P_{x\wedge y}(3) = \sum_{i=0}^{3}\sum_{j=0}^{3}\{P_x(i) \times P_y(j) \mid i \wedge j = 3\}$$

Since

$$P_{x\wedge y}(0) + P_{x\wedge y}(1) + P_{x\wedge y}(2) + P_{x\wedge y}(3) =$$

$$(P_x(0) + P_x(1) + P_x(2) + P_x(3)) \times (P_y(0) + P_y(1) + P_y(2) + P_y(3)) = 1$$

Similarly, probabilities for the formula $(x \vee y)$ and $(x \to y)$ are calculated [2].

The definition of a formula in the four-valued classical logic C4 is given inductively, so it is possible to calculate the probability of any formula in C4.

Conclusion. The created probabilistic four-valued logic can be applied to expert intellectual systems where the action of each factor on the system's operation can occur in four states (e.g., dominant, positive, neutral, or negative with a given probability)

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Transposed Poisson structures on Witt-type algebras

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Poisson algebras have appeared in several areas of mathematics and physics, such as algebraic geometry, operads, quantization theory, quantum groups, and classical and quantum mechanics. One of the natural tasks in the theory of Poisson algebras is the description of all such algebras with a fixed Lie or associative part. Recently, Bai, Bai, Guo, and Wu have introduced a dual notion of the Poisson algebra [1], called a transposed Poisson algebra, by exchanging the roles of the two multiplications in the Leibniz rule defining a Poisson algebra. In a recent paper by Ferreira, Kaygorodov, Lopatkin a relation between $\frac{1}{2}$-derivations of Lie algebras and transposed Poisson algebras has been established [2]. These ideas were used to describe all transposed Poisson structures for the several types of Lie algebras, for example, on Witt and Virasoro algebras, on Witt type Lie algebras, on generalized Witt algebras in [2, 3, 4].

The present work is a continuation of this research. Specifically, we describe transposed Poisson structures on the deformative Schrödinger-Witt algebra $\mathcal{W}(a,b,s)$. The algebras $\mathcal{W}(a,b,s)$ contain a subalgebra isomorphic to the well-known W-algebra $\mathcal{W}(a,b)$.

Definition 1. *Let $\mathfrak{L}$ be a vector space equipped with two nonzero bilinear operations $\cdot$ and $[\cdot,\cdot]$. The triple $(\mathfrak{L},\cdot,[\cdot,\cdot])$ is called a transposed Poisson algebra if $(\mathfrak{L},\cdot)$ is a commutative associative algebra and $(\mathfrak{L},[\cdot,\cdot])$ is a Lie algebra that satisfies the following compatibility condition*

$$2z\cdot[x,y]=[z\cdot x,y]+[x,z\cdot y].$$

Definition 2.*Let $(\mathfrak{L},[\cdot,\cdot])$ be a Lie algebra, $\varphi:\mathfrak{L}\to\mathfrak{L}$ be a linear map. Then φ is a $\frac{1}{2}$-derivation if it satisfies*

$$\varphi([x,y])=\tfrac{1}{2}\big([\varphi(x),y]+[x,\varphi(y)]\big).$$

The main example of $\frac{1}{2}$-derivations is the multiplication by an element from the ground field. Let us call such $\frac{1}{2}$-derivations as trivial $\frac{1}{2}$-derivations.

Lemma 1. [2] *Let $\mathfrak{L}$ be a Lie algebra without non-trivial $\frac{1}{2}$-derivations. Then every transposed Poisson structure defined on $\mathfrak{L}$ is trivial.*

Let $(\mathfrak{L}, \cdot)$ be an arbitrary commutative associative algebra, and let p be a fixed element of $\mathfrak{L}$. Then a new algebra is derived from $\mathfrak{L}$ by using the same vector space structure of $\mathfrak{L}$ but defining a new multiplication

$$x \star y = p \cdot x \cdot y$$

for $x, y \in \mathfrak{L}$. The resulting algebra is called the p-mutation of the algebra $\mathfrak{L}$.

Lemma 2. [2] *Let $(\mathfrak{L}, \cdot, [\cdot,\cdot])$ be a transposed Poisson algebra. Then every mutation of $(\mathfrak{L}, \cdot)$ gives a transposed Poisson algebra with the same Lie multiplication.*

Let us define the algebra $\mathcal{W}(a,b,s)$, where $a, b \in \mathbb{C}$ and $s \in \{0, \frac{1}{2}\}$. $\mathcal{W}(a,b,s)$ is spanned by $\{L_m, I_m, Y_{m+s} \mid m \in \mathbb{Z}\}$ and the multiplication table is given by the following nontrivial relations [5]:

$$\begin{array}{lcl} [L_m, L_n] & = & (n-m)L_{m+n}, \\ [L_m, I_n] & = & (n+bm+a)I_{m+n}, \\ [L_m, Y_{n+s}] & = & \left(n+s+\frac{(b-1)m+a}{2}\right)Y_{m+n+s}, \\ [Y_{m+s}, Y_{n+s}] & = & (n-m)I_{m+n+2s}. \end{array}$$

Since $\mathcal{W}(a+1,b,0) \cong \mathcal{W}(a,b,\frac{1}{2})$, we can only consider the case of $s = \frac{1}{2}$.

Proposition 1. If $b \neq -1$, then $\mathcal{W}(a,b,\frac{1}{2})$ does not have nontrivial $\frac{1}{2}$-derivations.

The following theorem gives the full description of nontrivial $\frac{1}{2}$-derivations of $\mathcal{W}(a,-1,\frac{1}{2})$.

Theorem 1. *Let φ be a $\frac{1}{2}$-derivation of the algebra $\mathcal{W}(a,-1,\frac{1}{2})$, then there are three sets of elements from the basic field $\{\alpha_t\}_{t\in\mathbb{Z}}$, $\{\beta_t\}_{t\in\mathbb{Z}}$ and $\{\gamma_t\}_{t\in\mathbb{Z}}$ such that*

$$\begin{array}{lcl} \varphi(L_m) & = & \sum\limits_{t\in\mathbb{Z}} \alpha_t L_{m+t} + \sum\limits_{t\in\mathbb{Z}} \beta_t I_{m+t} + \sum\limits_{t\in\mathbb{Z}} \gamma_t Y_{m+t+\frac{1}{2}}, \\ \varphi(I_m) & = & \sum\limits_{t\in\mathbb{Z}} \alpha_t I_{m+t}, \\ \varphi(Y_{m+\frac{1}{2}}) & = & \sum\limits_{t\in\mathbb{Z}} \alpha_t Y_{m+t+\frac{1}{2}} + \sum\limits_{t\in\mathbb{Z}} \gamma_t I_{m+t+1}. \end{array}$$

Now we consider the algebra $\mathcal{W}$ with a basis $\{L_i, I_i, Y_{i+\frac{1}{2}} \mid i \in \mathbb{Z}\}$ given by the following commutative multiplication table:

$$L_i L_j = L_{i+j}, \quad L_i I_j = I_{i+j}, \quad L_i Y_{j+\frac{1}{2}} = Y_{i+j+\frac{1}{2}}, \quad Y_{i+\frac{1}{2}} Y_{j+\frac{1}{2}} = I_{i+j+1}.$$

In the following, we aim to classify all transposed Poisson structures on $\mathcal{W}(a,-1,\frac{1}{2})$.

Theorem 2. *Let* $(\mathfrak{L},\cdot,[\cdot,\cdot])$ *be a transposed Poisson structure defined on the Lie algebra* $\mathcal{W}(a,-1,\frac{1}{2})$*. Then* $(\mathfrak{L},\cdot,[\cdot,\cdot])$ *is not Poisson algebra and* $(\mathfrak{L},\cdot)$ *is a mutation of the algebra* $\mathcal{W}$*. On the other hand, every mutation of the algebra* $\mathcal{W}$ *gives a transposed Poisson structure with the Lie part isomorphic to* $\mathcal{W}(a,-1,\frac{1}{2})$*.*

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Embankment surface according to bishop frame in euclidean 3-space

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In this presentation, we introduce embankment, embankmentlike and tubembankmentlike surfaces determined by regular space curve according to Bishop Frame in Euclidean $3-$space. We give parametric equations, matrix and quaternionic representations of them. Later, we give some geometric properties of them. Then, we give some theorems and conclusions related with these surfaces and parameter curves of them. At the end of this work, we give some related examples about each surface with their figures.

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Compactness of weighted composition operator

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Let us introduce some needful information which will be used in this report.

A topological space X is said to be **completely regular** if for arbitrary x in X and any closed subset A of X not containing x there exist a continuous function $f : X \to [0,1]$ such that $f(x) = 1$ and $f(y) = 0$ for all y in A. A **Banach function algebra on a compact Hausdorff space** X is a Banach algebra A consisting of continuous functions on X such that A separates points of X and contains the constant functions. A **uniform algebra** on a compact Hausdorff space X is a uniformly closed subalgebra of $C(X)$, which contains the constants and separates the points of X. Two points $S_0(A(X))$ of a topological space $S_0(A(X))$ are called **compactly connected** if there exists a connected compact set $S(A(X))$ such that it contains both of these points. A subset K of X is called **a peak set** for A if there is an $f \in A$ such that $f \equiv 1$ on K and $|f(x)| < 1$ for $x \notin K$. If_$K = \{x_0\}$, then x_0 is called a peak point. Let $A(X)$ be a uniformly closed subspace of $C(X)$ (in particular, a uniform algebra). A closed subset $E \subset X$is called **a peak set with respect to** $A(X)$**,** if there exists a sequence $\{f_n\} \subset A(X)$ such that $\|f_n\| = f_n(x) = 1$ for all n and all $x \in E$. Moreover, outside any neighborhood of the set E the sequence $\{f_n\}$ tends to 0uniformly. A peak set consisting of only one point is called **a peak point with respect to** $A(X)$. Denote the set of all peak sets with respect to $A(X)$ by $S(A(X))$ and the set of all peak points with respect to $A(X)$ by $S_0(A(X))$. We will assume that the set $S_0(A(X))$ is dense in $S(A(X))$ and the number of the compactly connected components of $S(A(X))$ is finite. Let $A(X)$ be a uniformly closed subspace of $C(X)$ (in particular, a uniform algebra). A mapping $\varphi : X \to X$ is called a **compositor on** $A(X)$**,** if $f \circ \varphi \in A(X)$ whenever$f \in A(X)$. A function $u \in C(X)$ is called a **multiplicator with respect to** $A(X)$ if $u \cdot f \in A(X)$

for all functions $f \in A(X)$. We denote the set of all compositors on $A(X)$ by $C_{A(X)}$, and the set of all multiplicators with respect to $A(X)$ by $M_{A(X)}$.

We will only consider the following case: let $C(X)$ be a Banach algebra of continuous functions, $A = A(X) \subset C(X)$ be a subalgebra, and $C(X, A)$ be the algebra of A-valued continuous functions on X with the norm $\|h\|_{C(X.A)} = \sup\limits_{x \in X} \|h(x)\|_{C(X)}$. Also, let $\varphi : X \to X$ be a self-mapping of X, and $u \in C(X, A)$ be a given function. Consider the weighted composition operator $T : C(X, A) \to C(X, A)$ defined as follows:

$$(T \cdot f)(x) = u(f \circ \varphi)(x), f \in C(X, A), u \in M_{A(X)}, \varphi \in C_{A(X)}.$$

In the sequel, by $s(u)$ we will denote$s(u) = \{x \in X : \exists u^{-1}(x)\}$.

Convention. We assume that$s(u) = \{x : u(x) \neq 0\}$, i.e.

$$\forall x \in X \Rightarrow u(x) = 0 \vee \exists u^{-1}(x).$$

We will also assume that $\varphi\big|_{s(u)}$ is a continuous function on $s(u)$. From classical facts it follows that $s(u)$ is an open set of$X$.

The following statements are obtained.

Lemma 1. *Let X, Y be a compact metric spaces, S be a metric space, and $\varphi : X \to Y$ be a continuous mapping from X on Y. Then the family $F \subset C(Y, S)$ is uniformly equicontinuous if and only if the family $F \circ \varphi = \{f \circ \varphi : f \in C(Y)\}$ is uniformly equicontinuous in $C(X, S)$.*

Lemma 2. *Let X, Y be a compact metric spaces, $A(X)$ be some uniform algebra and $\varphi : X \to X$be a self-mapping of X, which is continuous on $s(u)$. The weighted composition operator T is compact if and only if for an arbitrary compact subset $K \subset s(u)$ the restriction of the family $U = \left\{f \in C(X, A) : \|f\|_{C(X,A)} \leq 1\right\}$ to the compact $\varphi(K)$ is an equicontinuous family.*

Theorem. *Let the operator T defined by be a compact operator. Then for any compactly connected component $K \subset s_u$ and for any peak set E with respect to $A(X)$, we have either $\varphi(K) \subseteq E$, or $\varphi(K) \bigcap E = \emptyset$.*

On the factorization problem of groups and monoids

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The problem of factorization of mathematical objects has a long history. In order to give a solution to this problem for monoids and groups, we introduced the concepts of *left* and right *quasi-projectors* on a monoid in [1]. In this talk I will explain how they can be combined with the results of [2] and [3] to provide a complete classification of factorizations of monoids in terms of *complementary pairs* of quasi-projections.

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Holomorphic manifolds and problems of lifts

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Our goal of this presentation is to expound the recent developments in applications of holomorphic functions in the theory of hypercomplex (Section I) and anti-Hermitian manifolds (Section II) as well as in the geometry of bundles (Section III).

In Section I we give the fundamental notions and some theorems concerning holomorphic functions and holomorphic manifolds which will be needed for the later applications. In Section 1.1, we give the basic definitions and theorems on hypercomplex algebras. Section 1.2 is devoted to the study of holomorphic functions in algebra. In Section 1.3, we introduce the hypercomplex structures on manifolds. Section 1.4 treats manifolds with integrable regular hypercomplex structures. We show that such a manifold is a realization of a holomorphic manifold over algebra. Section 1.5 is devoted to the study of pure tensor fields. We find the explicit expression of the pure tensor field with respect to the regular hypercomplex structures and we show that the pure tensor fields on real manifolds are a realization of hypercomplex tensors. In Section 1.6, we discuss holomorphic hypercomplex tensor fields and using the Tachibana operator we give the condition of holomorphic tensors in real coordinates. In Section 1.7, we consider pure connections which are realizations of the hypercomplex connections. Section 1.8 is devoted to the study of pure hypercomplex torsion tensors. In Section 1.9, we give a realization of holomorphic hypercomplex connections by using the pure curvature tensors. In the last Section 1.10, we consider some properties of pure curvature tensors. The main theorem of this section is that the curvature tensor of a holomorphic connection is holomorphic. Finally, we also consider a holomorphic manifold of hypercomplex dimension 1 and we show that the hypercomplex connection on such a manifold is holomorphic if and only if the real manifold is locally flat.

In Section II we study the pseudo-Riemannian metric on holomorphic manifolds. In Section 2.1, we give the condition for a hypercomplex anti-Hermitian metric to be holomorphic, also we prove that there exists a one-to-one correspondence between hypercomplex anti-Kähler manifolds and anti-Hermitian manifolds with an $\mathcal{A}$-holomorphic metrics. In Section 2.2, we discuss complex Norden manifolds. We define the twin Norden metric, the main theorem of this section is that the Levi-Civita connection of Kähler-Norden metric coincides with the Levi-Civita connection of twin Norden metric. In Section 2.3, we consider Norden-Hessian structures. We give the condition for a Norden-Hessian manifold to be Kähler. Section 2.4 is devoted to the analysis of twin Norden metric connections with torsion. In Sections 2.5 - 2.10, we focus our attention to pseudo-Riemannian 4-manifolds of neutral signature. The main purpose of these sections is to study complex Norden metrics on 4-dimensional Walker manifolds. We discuss the integrability and Kahler (holomorphic) conditions for these structures. The curvature properties for Norden-Walker metrics is also investigated and examples of Norden-Walker metrics are constructed from an arbitrary harmonic function of two variables. We define the isotropic Kähler structures and moreover, show that a proper almost complex structure on an almost Norden-Walker manifold is isotropic Kähler. We also consider the quasi-Kähler-Norden metric and give the condition for an almost Norden manifold to be quasi-Kähler-Norden. Finally we give progress to the conjecture of Goldberg under the additional restriction on Norden-Walker metric.

In the first part of Section III we focus on lifts from a manifold to its tensor bundle. Some introductory material concerning the tensor bundle is provided in Section 3.1. Section 3.2 is devoted to the study of the complete lifts of (1,1)-tensor fields along cross-setions in the tensor bundle. In Section 3.3 we study holomorphic cross-sections of tensor bundles. In the second part we concentrate our attention to lifts from a manifold to its tangent bundles of order 1 and 2 by using the realization of holomorphic manifolds. The main purpose of Sections 3.4-3.9 is to study the differential geometrical objects on the tangent bundle of order 1 corresponding to dual-holomorphic objects of the dual-holomorphic manifold. As a result of this approach, we find a new class of lifts, i.e. deformed complete lifts of functions, vector fields, forms, tensor fields and linear connections in the tangent bundle of order 1. Section 3.10 is devoted to the study of holomorphic metrics in the tangent bundle of order 2 (i.e. in the bundle of 2-jets) by using the Tachibana operator. By using the algebraic approach, the problem of deformed lifts of functions, vector fields and 1-forms is solved in Sections 3.11-3.12. In Section 3.13, we investigate the complete lift of the almost complex structure to cotangent bundle and prove that it is a transfer by a symplectic isomorphism of complete lift to tangent bundle if the symplectic manifold with almost complex structure is an almost holomorphic A-manifold. Finally, in Section 3.14 we transfer via the differential of the musical isomorphism defined by pseudo-Riemannian metrics the complete lifts of vector fields and almost complex structures from the tangent bundle to the cotangent bundle.

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Some combinatorial identities for higher order Fibonacci numbers via the Toeplitz–Hessenberg determinants

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Many generalizations of Fibonacci numbers F_n (sequence A000045 in [3]) have been examined by many mathematicians in the literature. One of these generalizations is the *Fibonacci divisor*, also known as *higher order Fibonacci numbers* [4]. These numbers are defined as follows: $F_n^{(s)} = \frac{F_{ns}}{F_s}$, $n \geq 0$, $s \geq 1$. Due to F_{ns} being divisible by F_s, the ratio F_{ns}/F_s is an integer. For $s = 1$, we have $F_n^{(1)} = F_n$.

A Toeplitz–Hessenberg matrix is an $n \times n$ matrix of the form

$$M_n(a_0; a_1, \ldots, a_n) = \begin{pmatrix} a_1 & a_0 & 0 & \cdots & 0 & 0 \\ a_2 & a_1 & a_0 & \cdots & 0 & 0 \\ \ldots & \ldots & \ldots & \ddots & \ldots & \ldots \\ a_{n-1} & a_{n-2} & a_{n-3} & \cdots & a_1 & a_0 \\ a_n & a_{n-1} & a_{n-2} & \cdots & a_2 & a_1 \end{pmatrix},$$

where $a_0 \neq 0$ and $a_k \neq 0$ for at least one $k > 0$.

We investigate some families of Toeplitz–Hessenberg determinants with $a_0 = \pm 1$ the entries of which are higher order Fibonacci numbers. Using the Trudi formula (see [1, 2] for more details), these determinant formulas may be rewritten as identities involving sums of products of the Fibonacci numbers and multinomial coefficients.

For brevity, we will write $D_{\pm}(a_1, \ldots, a_n)$ instead of $\det\big(M_n(\pm 1; a_1, \ldots, a_n)\big)$.

The following theorems give the value of $D_{\pm 1}(a_1, \ldots, a_n)$ for some entries $F_i^{(2)} = F_{2i}$, $F_i^{(3)} = \frac{1}{2} F_{3i}$, and $F_i^{(4)} = \frac{1}{3} F_{4i}$.

Theorem 2. *Let $n \geq 1$, except when noted otherwise. Then*

$$D_-(F_0^{(2)}, F_1^{(2)}, \ldots, F_{n-1}^{(2)}) = 3^{n-2}, \qquad n \geq 2;$$
$$D_-(F_1^{(2)}, F_2^{(2)}, \ldots, F_n^{(2)}) = \frac{\sqrt{3}}{6}\Big((2+\sqrt{3})^n - (2-\sqrt{3})^n\Big);$$
$$D_-(F_2^{(2)}, F_3^{(2)}, \ldots, F_{n+1}^{(2)}) = \frac{\sqrt{7}}{28}\Big((3+\sqrt{7})^{n+1} - (3-\sqrt{7})^{n+1}\Big);$$
$$D_-(F_1^{(2)}, F_3^{(2)}, \ldots, F_{2n-1}^{(2)}) = 9 \cdot 8^{n-2}, \qquad n \geq 2;$$
$$D_+(F_3^{(2)}, F_5^{(2)}, \ldots, F_{2n+1}^{(2)}) = 9, \qquad n \geq 2;$$
$$D_+(F_2^{(2)}, F_4^{(2)}, \ldots, F_{2n}^{(2)}) = \frac{\sqrt{3}}{2}\Big((-2+\sqrt{3})^n - (-2-\sqrt{3})^n\Big).$$

Theorem 3. *Let $n \geq 1$, except when noted otherwise. Then*

$$D_+(F_0^{(3)}, F_1^{(3)}, \ldots, F_{n-1}^{(3)}) = (-1)^{n-1}4^{n-2}, \qquad n \geq 2;$$
$$D_-(F_0^{(3)}, F_1^{(3)}, \ldots, F_{n-1}^{(3)}) = \frac{\sqrt{6}}{12}\Big((2+\sqrt{6})^{n-1} - (2-\sqrt{6})^{n-1}\Big);$$
$$D_+(F_2^{(3)}, F_3^{(3)}, \ldots, F_{n+1}^{(3)}) = 0, \qquad n \geq 3;$$
$$D_-(F_2^{(3)}, F_3^{(3)}, \ldots, F_{n+1}^{(3)}) = \frac{\sqrt{2}}{24}\Big((4+3\sqrt{2})^{n+1} - (4-3\sqrt{2})^{n+1}\Big);$$
$$D_+(F_1^{(3)}, F_3^{(3)}, \ldots, F_{2n-1}^{(3)}) = (-1)^{n-1}16 \cdot 17^{n-2}, \qquad n \geq 2;$$
$$D_+(F_3^{(3)}, F_5^{(3)}, \ldots, F_{2n+1}^{(3)}) = (-1)^{n-1}16, \qquad n \geq 2.$$

Theorem 4. *Let $n \geq 1$, except when noted otherwise. Then*

$$D_-(F_0^{(4)}, F_1^{(4)}, \ldots, F_{n-1}^{(4)}) = 7^{n-2}, \qquad n \geq 2;$$
$$D_+(F_1^{(4)}, F_2^{(4)}, \ldots, F_n^{(4)}) = \frac{\sqrt{2}}{8}\Big((-3+2\sqrt{2})^n - (-3-2\sqrt{2})^n\Big);$$
$$D_-(F_1^{(4)}, F_2^{(4)}, \ldots, F_n^{(4)}) = \frac{\sqrt{15}}{30}\Big((4+\sqrt{15})^n - (4-\sqrt{15})^n\Big);$$
$$D_+(F_2^{(4)}, F_3^{(4)}, \ldots, F_{n+1}^{(4)}) = 0, \qquad n \geq 3;$$
$$D_-(F_1^{(4)}, F_3^{(4)}, \ldots, F_{2n-1}^{(4)}) = 49 \cdot 48^{n-2}, \qquad n \geq 2;$$
$$D_+(F_3^{(4)}, F_5^{(4)}, \ldots, F_{2n+1}^{(4)}) = 49, \qquad n \geq 2.$$

Now from Theorems 2-4, using Trudi's formula, we obtain identities involving Fibonacci numbers and multinomial coefficients $m_n(t) = \frac{(t_1+\cdots+t_n)!}{t_1!\cdots t_n!}$. We will provide only a few such formulas.

Corollary 5. *The following identities hold*:

$$\sum_{\sigma_n=n} m_n(t) F_2^{t_1} F_6^{t_2} \cdots F_{2(2n-1)}^{t_n} = 9 \cdot 8^{n-2}, \qquad n \geq 1$$

$$\sum_{\sigma_n=n} (-1)^{|t|} m_n(t) F_6^{t_1} F_{10}^{t_2} \cdots F_{2(2n+1)}^{t_n} = (-1)^n 9, \qquad n \geq 2,$$

$$\sum_{\sigma_n=n} \left(-\frac{1}{2}\right)^{|t|} m_n(t) F_6^{t_1} F_9^{t_2} \cdots F_{3(n+1)}^{t_n} = 0, \qquad n \geq 3,$$

$$\sum_{\sigma_n=n} \left(-\frac{1}{2}\right)^{|t|} m_n(t) F_3^{t_1} F_9^{t_2} \cdots F_{3(2n-1)}^{t_n} = -16 \cdot 17^{n-2}, \qquad n \geq 2,$$

$$\sum_{\sigma_n=n} \left(-\frac{1}{3}\right)^{|t|} m_n(t) F_8^{t_1} F_{12}^{t_2} \cdots F_{4(n+1)}^{t_n} = 0, \qquad n \geq 3,$$

$$\sum_{\sigma_n=n} \left(-\frac{1}{3}\right)^{|t|} m_n(t) F_{12}^{t_1} F_{20}^{t_2} \cdots F_{4(2n+1)}^{t_n} = (-1)^n 49, \qquad n \geq 2,$$

where $|t| := t_1 + \cdots + t_n$ *and* $\sigma_n := t_1 + 2t_2 + \cdots + nt_n$ *with* $t_i \geq 0$.

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On the weak-injectivity domains of modules

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There is a new trend in Module Theory which expands the study of module properties by developing mechanisms. This approach allows for the consideration of not only modules that satisfy a property, but also those that partially or even minimally satisfy it. Injective modules are among the most important homological objects in module categories. They play a significant role in both Module and Ring Theory, as well as in Homological Algebra. Determining whether a module is injective can be quite difficult. Recently, the concept of the injectivity domain has gained increasing interest as an idea to measure the injectivity of a module. The class of modules to which M is relatively injective is said to be *the injectivity domain of* M. The measuring tools for modules in this mechanism are portfolios. If a class of modules is an injectivity domain of a module, that class is called an (*injective*) *portfolio*, and the class of all possible (injective) portfolios for all right R-modules over a ring R is called the (*injective*) *profile* of R.

Motivated by the fact that a ring R is right Noetherian if and only if direct sums of injective modules are injective, López-Permouth and Saraç considered this in terms of portfolios. The idea was when a module added to a poor module it remains poor. In [4] López-Permouth and Saraç considered portfolios in the injective profile of a ring R as layers and they asked the question that is there an injective portfolio in the middle of the injective profile of the ring R, where every layer below it becomes stable? If the injectivity domain of the direct sum of every module family whose injectivity domain is a portfolio $\mathcal{A}$ is also $\mathcal{A}$, then $\mathcal{A}$ is called a *stable portfolio.* The answer to this question is yes, and this injective portfolio is called the *Noetherian threshold.* It is well-known that the injectivity of certain modules determines the ring as Noetherian (see [3]). In [4] the injectivity domain of these certain modules have used to measure the extent to which a ring is Noetherian. Additionally, as an opposite of Noetherianness of a ring they defined *volatile rings.* These are the rings in which only poor modules have stable injective portfolios (see [4]).

It is tempting to think may be such analysis can be run on something else. The notion of weak-injectivity can play a similar role to the concept of injectivity in order to characterize when direct sums of injectives are weakly-injective. In [1], it was shown that a ring is a q.f.d. ring if and only if the direct sums of injective modules are weakly-injective. In this work (small) weak-injectivity domains of modules, which have defined in [2] to measure the scope of injectivity or weak-injectivity, are used to measure the extent to which a ring is q.f.d. It is explored that whether there is a (small) weakly-injective portfolio in the middle of the (small) weak-injectivity profile of a ring R, where every layer below it is stable. Also volatile rings are defined in conjunction with weak-injectivity, their properties are investigated, and examples are provided. Furthermore, the volatility of a ring using the weak-injectivity profile is examined.

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Mappings on neutrosophic topology on soft sets

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The present study is devoted to describe the concepts of continuous mapping, open mapping and closed mapping by using soft points on neutrosophic topological spaces. Along, continuous mapping, open mapping and closed mapping on neutrosophic topological spaces and their characterizations are also introduced. At the end, some of the crucial properties of the proposed concepts are investigated. Taking advantage of neutrosophic topology, we obtain the family of soft topologies and normal topologies. It is clear that the category of neutrosophic topological spaces is an extension both the category of soft topological spaces and the category of topological spaces.

Mapping on Neutrosophic Topology on Soft Sets

Definition 1. *Let (X, E, τ) and (Y, E', γ) be two neutrosophic topological spaces and $(f, \varphi) : (X, E, \tau) \to (Y, E', \gamma)$ be a mapping. Then (f, φ) is called a continuous mapping at the soft point $x_e \in (X, E)$ if there exists $x_e \in (F, E) \in SS(X, E)$ such that $\tau_T(F, E) \geq \gamma_T(G, E')$, $\tau_I(F, E) \geq \gamma_I(G, E')$, $\tau_F(F, E) \leq \gamma_F(G, E')$ and $(f, \varphi)(F, E) \subset (G, E')$ for each arbitrary soft set $(f, \varphi)(x_e) = (f(x))_{\varphi(e)} \in (G, E') \in SS(Y, E')$. If (f, φ) is a continuous mapping for each soft point, then (f, φ) is a continuous mapping.*

Theorem 1. *Let (X, E, τ) and (Y, E', γ) be two neutrosophic topological spaces and $(f, \varphi) : (X, E, \tau) \to (Y, E', \gamma)$ be a mapping. Then (f, φ) is a continuous mapping if and only if $\tau_T\left((f, \varphi)^{-1}(G, E')\right) \geq \gamma_T(G, E')$, $\tau_I\left((f, \varphi)^{-1}(G, E')\right) \geq \gamma_I(G, E')$, $\tau_F\left((f, \varphi)^{-1}(G, E')\right) \leq \gamma_F(G, E')$ are satisfied for each $(G, E') \in SS(Y, E')$.*

Theorem 2. *Let (X, E, τ) and (Y, E', γ) be two neutrosophic topological spaces and $(f, \varphi) : (X, E, \tau) \to (Y, E', \gamma)$ be a mapping. Then (f, φ) is a continuous mapping if and only if $(f_r, \varphi_r) : (X, E, \tau_r) \to (Y, E', \gamma_r)$ is a continuous mapping on soft tritopological space for each $r \in (0, 1]$.*

Theorem 3. *Let (X, E, τ) and (Y, E', γ) be two neutrosophic topological spaces and $(\beta_T, \beta_{\mathrm{I}}, \beta_F)$ be a base of $(\gamma_T, \gamma_{\mathrm{I}}, \gamma_F)$ on Y. Then $(f, \varphi) : (X, E, \tau) \to$*

(Y,E',γ) *is a continuous mapping if and only if*

$$\begin{array}{l}\beta_T\left(G,E'\right)\le\tau_T\left(\left(f,\varphi\right)^{-1}\left(G,E'\right)\right),\\ \beta_I\left(G,E'\right)\le\tau_I\left(\left(f,\varphi\right)^{-1}\left(G,E'\right)\right),\\ \beta_F\left(G,E'\right)\ge\tau_{IF}\left(\left(f,\varphi\right)^{-1}\left(G,E'\right)\right),\end{array}$$

for each $(G,E')\in SS\left(Y,E'\right)$.

Theorem 4. *Let* (X,E,τ) *and* (Y,E',γ) *be two neutrosophic topological spaces and* $(\delta_T,\delta_{\mathrm{I}},\delta_F)$ *be a subbase of* $(\gamma_T,\gamma_{\mathrm{I}},\gamma_F)$. *If*

$$\begin{array}{l}\delta_T\left(G,E'\right)\le\tau_T\left(\left(f,\varphi\right)^{-1}\left(G,E'\right)\right),\\ \delta_I\left(G,E'\right)\le\tau_I\left(\left(f,\varphi\right)^{-1}\left(G,E'\right)\right),\\ \delta_F\left(G,E'\right)\ge\tau_F\left(\left(f,\varphi\right)^{-1}\left(G,E'\right)\right),\end{array}$$

are satisfied for each $(G,E')\in SS\left(Y,E'\right)$, *then* $(f,\varphi):(X,E,\tau)\to(Y,E',\gamma)$ *is a continuous mapping.*

Definition 2. *Let* (X,E,τ) *and* (Y,E',γ) *be two neutrosophic topological spaces and* (f,φ) *be a mapping from* (X,E,τ) *to* (Y,E',γ). *The mapping* (f,φ) *is called an open mapping if it satisfies the following condition:*

$$\begin{array}{l}\tau_T\left(F,E\right)\le\gamma_T\left(\left(f,\varphi\right)\left(F,E\right)\right),\\ \tau_I\left(F,E\right)\le\gamma_I\left(\left(f,\varphi\right)\left(F,E\right)\right),\\ \tau_F\left(F,E\right)\ge\gamma_F\left(\left(f,\varphi\right)\left(F,E\right)\right)\end{array}$$

for each $(F,E)\in SS\left(X,E\right)$.

Theorem 5. *Let* (X,E,τ) *and* (Y,E',γ) *be two neutrosophic topological spaces and* $(f,\varphi):(X,E,\tau)\to(Y,E',\gamma)$ *be a mapping and* $(\beta_T,\beta_{\mathrm{I}},\beta_F)$ *be a base of* (τ_T,τ_I,τ_F). *If*

$$\begin{array}{l}\beta_T\left(F,E\right)\le\gamma_T\left(\left(f,\varphi\right)\left(F,E\right)\right),\\ \beta_I\left(F,E\right)\le\gamma_I\left(\left(f,\varphi\right)\left(F,E\right)\right),\\ \beta_F\left(F,E\right)\ge\gamma_F\left(\left(f,\varphi\right)\left(F,E\right)\right),\end{array}$$

are satisfied for each $(F,E)\in SS\left(X,E\right)$, *then* (f,φ) *is an open mapping.*

Theorem 6. *Let* (Y,E',γ) *be a neutrosophic topological space and* $(f,\varphi):SS\left(X,E\right)\to(Y,E',\gamma)$ *be a mapping of soft sets. Then define* $(\tau_T,\tau_I,\tau_F):SS\left(X,E\right)\to[0,1]$ *by:*

$$\begin{array}{l}\tau_T\left(F,E\right)=\bigvee\limits_{f^{-1}(G,E')=(F,E)}\gamma_T\left(G,E'\right),\\ \tau_I\left(F,E\right)=\bigvee\limits_{f^{-1}(G,E')=(F,E)}\gamma_I\left(G,E'\right),\\ \tau_F\left(F,E\right)=\bigwedge\limits_{f^{-1}(G,E')=(F,E)}\gamma_F\left(G,E'\right),\end{array}$$

for each $(F,E) \in SS(X,E)$. *Then* (τ_T, τ_I, τ_F) *is a neutrosophic topology on* X *and* (f,φ) *is a continuous mapping.*

Theorem 7. *Let* (X,E,τ) *be a neutrosophic topological space and* $(f,\varphi):(X,E,\tau)\to SS(Y,E')$ *be a mapping of soft sets. Then define* $(\gamma_T,\gamma_I,\gamma_F): SS(Y,E')\to[0,1]$ *by:*

$$\gamma_T(G,E') = \tau_T\left((f,\varphi)^{-1}(G,E')\right),\ \gamma_I(G,E') = \tau_I\left((f,\varphi)^{-1}(G,E')\right),$$

$$\gamma_F(G,E') = \tau_F\left((f,\varphi)^{-1}(G,E')\right),$$

for each $\forall(G,E')\in SS(Y,E')$. *Then* $(\gamma_T,\gamma_{\mathrm{I}},\gamma_F)$ *is a neutrosophic topology on* Y *and* (f,φ) *is a continuous mapping.*

Theorem 8. *Let* $\{(X_\lambda,E_\lambda,\tau_\lambda)\}_{\lambda\in\Lambda}$ *be a family of neutrosophic topological spaces,* $X=\prod_{\lambda\in\Lambda}X_\lambda$ *be a set,* $E=\prod_{\lambda\in\Lambda}E_\lambda$ *be a parameter set and for each* $\lambda\in\Lambda$, $\rho_\lambda: X\to X_\lambda$ *and* $q_\lambda: E\to E_\lambda$ *be two projections maps. Define* $(\beta_T,\beta_{\mathrm{I}},\beta_F): SS(Y,E')\to[0,1]$ *by:*

$$\beta_T(G,E') = \vee\left\{\overset{n}{\underset{j=1}{\wedge}}\tau_{T_{\alpha_j}}\left(F_{\alpha_j},E_{\alpha_j}\right):(F,E)=\bigcap_{j=1}^{n}\left(\rho_{\alpha_j},q_{\alpha_j}\right)^{-1}\left(F_{\alpha_j},E_{\alpha_j}\right)\right\},$$
$$\beta_I(G,E') = \vee\left\{\overset{n}{\underset{j=1}{\wedge}}\tau_{I_{\alpha_j}}\left(F_{\alpha_j},E_{\alpha_j}\right):(F,E)=\bigcap_{j=1}^{n}\left(\rho_{\alpha_j},q_{\alpha_j}\right)^{-1}\left(F_{\alpha_j},E_{\alpha_j}\right)\right\},$$
$$\beta_F(G,E') = \wedge\left\{\overset{n}{\underset{j=1}{\vee}}\tau_{F_{\alpha_j}}\left(F_{\alpha_j},E_{\alpha_j}\right):(F,E)=\bigcap_{j=1}^{n}\left(\rho_{\alpha_j},q_{\alpha_j}\right)^{-1}\left(F_{\alpha_j},E_{\alpha_j}\right)\right\}$$

Then $(\beta_T,\beta_{\mathrm{I}},\beta_F)$ *is a base on NTS and* $(\rho_\lambda,q_\lambda):(X,E,\tau_\beta)\to(X_\lambda,E_\lambda,\tau_\lambda)$ *are continuous maps for each or each* $\lambda\in\Lambda$.

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ASTRONOMY

Character of changes in hydrogen lines in the spectra magnetic – CP stars εUMa and θAur.

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The study of hydrogen lines in the spectra of normal and magnetic chemically peculiar (MCP) stars is an urgent problem, as it provides important information about the structure and dynamics of stellar atmospheres. This paper presents the results of a comparison of the observed contours of hydrogen lines (H lines) of two magnetic stars εUMa and θAur. These stars are representatives of different types of peculiarity (Cr, Eu and Si, respectively) of magnetic stars, and are included in the catalog of N. Tusi (in the Elkhan table).

A large number of spectra were obtained for these stars, which cover the full rotation period of each star. All spectra were obtained at the 2 m telescope of the Shamakhi Astrophysical Observatory using echelle spectrometers installed at the kude and Cassegrain focus with a resolution of R=30000 and R=14000 [1, 2]. The observational material was processed using the DECH software package. All kude spectrograms were obtained with high time resolutions - 1.5-2 minutes. These observational materials were also used to determine short-period changes in the stars under study.

In [3] it was noted that the star εUMa has short-period (6-18 minutes) [4] variability in the intensity of hydrogen lines. Despite the fact that spectrograms with high temporal resolution were used, we were unable to detect quasi-periodic changes in this star.

The fact is that quasi-periodic changes are difficult to detect by the spectrophotometric method due to its low accuracy. In addition, quasi-periodic changes do not occur in all phases, but only in the phase of the corresponding peculiar region on the surface of the MCP-stars.

Detailed information about the star εUMa was presented in [5].

The aim of this work is to identify possible differences in the nature of changes in hydrogen (H) lines in the spectra of different types of εUMa (Cr, Eu) and θAur (Si) peculiar magnetic stars.

In practically all the spectra of magnetic-CP stars, the intensities of the H-lines change during the period. According to the work [6], in some magnetic stars, the greatest changes are found in the higher members (n≥10, H10-H16) of the Balmer series, and in others in the lower members (H_α-H_ε) of the series. For this purpose, the equivalent widths (W_λ) of the H5-H14 lineages in the εUMa and θAur spectra were determined. The contours of the H_α and H_β lines in the spectra of the studied stars have peculiarities: the central parts of the lines are narrower, and the wings are somewhat wider than those of the normal stars θLeo and αLyr. For the higher members (n≥15) of the Balmer series, conducting a continuous spectrum is not very confident due to the overlapping lines. Therefore, we did not measure the equivalent widths of these lines.

According to the purpose of this work, we calculated the relative values of equivalent widths ($W_\lambda/\overline{W_\lambda}$) H – lines. Amplitudes of relative values of equivalent widths, - $\delta A =(W_\lambda/\overline{W_\lambda})_{max}$ - $(W_\lambda/\overline{W_\lambda})_{min}$, for each selected H line (H5-H16) were found. Next, the dependencies of the amplitude (δA) and the change in values ($W_\lambda/\overline{W_\lambda}$) on the wavelength were plotted (see Fig. 1). From Fig shows that the value of δA for εUMa decreases with wavelength, and in the case of θAur, the opposite pattern is observed, that is, δA increases with wavelength.

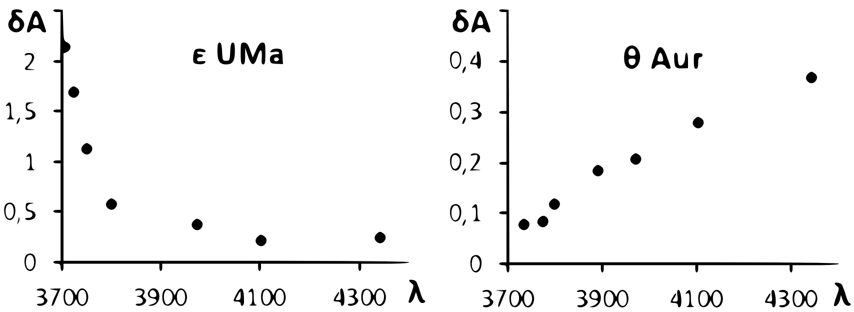

Figure 1: Dependence of amplitude δA on wavelength for stars εUMa and θAur

It is known that the effective depths of formation (τ) of the Balmer lines vary greatly, with high-numbered hydrogen lines forming in the uppermost layers of the atmosphere with $\tau_\lambda \sim$ 0.1-0.2, and H_γ-H_ε lines forming in the deep layers with $\tau_\lambda \sim$ 0.5-0.7 for stars of the B2-F0 classes [7].

From Fig. 1, it is clear that for the star εUMa the greatest changes are shown by lines with high numbers, and for θAur – by lines with low numbers (H_γ-H_ε). It follows that peculiar layers (peculiarities), especially chemical anomalies (which are responsible for changes in spectral lines) are located at different optical depths - τ. This means that the peculiar layers are located in the upper layers ($\tau \approx 0.2$) of the atmosphere of the star εUMa, and for θAur - in the relatively lower layers ($\tau \approx 0.6$) of the atmosphere.

Based on the above facts, the following conclusions can be drawn:

1. For the star εUMa, the peculiarity decreases with atmospheric depth.
2. But, for the star θAur peculiarity increases with the depth of the atmosphere.

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The significance of Nasiraddin Tusi's works for the development of celestial mechanics and astronomy

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The report provides an overview of scientific ideas and their material embodiment by an outstanding scientist of medieval civilization, Nasiraddin Tusi, whose kinematic studies and modeling of the motion of planets and celestial bodies gave new insights into the nature of motion in space and helped to form the modern understanding of kinematics and astronomy. The result of Nasiraddin Tusi's scientific and astronomical research conducted at the observatory in Maragha was the publication of the work "Astronomical Tables of Ilkhani". This work presented the main elements of the geocentric orbits of the planets, their average rotation per day, which turned out to be more accurate than astronomical studies of the XVII century. In addition, the "Astronomical Tables of Ilkhani" contained many mathematical, astronomical and geographical tables. With this work, Tusi inscribed himself into the world astronomical science.

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COMPUTATIONAL MATHEMATICS AND MATHEMATICAL MODELING

A high order finite volumes strategy for the solution of non-linear elasticity

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The objective of this work is to study the propagation of cracks in a hyperelastic material. To do so, one needs to evaluate, with high accuracy, the stresses in the vicinity of the crack. Only High Order Schemes, with good local conservation properties such the Finite Volumes Method, are able to provide this kind of an answer.

After a detailed presentation of the mathematical model, we will discuss in detail the proposed high-order finite volume method and give error estimates as well as numerical tests, showing the effectiveness of the proposed method.

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About one class of nonlinear optimization problems large dimension

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We consider an object of complex structure, consisting of a large number N of subobjects interconnected in random order. The states of subobjects depend on a control object consisting of M independent control subobjects. The state of the i-th subobject, determined by the n-dimensional vector $z_i \in R^{n_i}$, is influenced by the values of the μ_i-dimensional vector of its internal parameter $p_i \in R^{\mu_i}$, $i = 1,..,N$, the states of adjacent subobjects z_j, $j \in I_i \subset I = \{1,..,N\}$ and the r_s-dimensional states of independent control subobjects $u_s \in R^{r_s}$, $s \in J_i \subset J = \{1,..,M\}$, $i = 1,..,N$.

Let us introduce the following notations:

$$I = \bigcup_{i=1}^{N} I_i, \quad J = \bigcup_{i=1}^{M} J_i, \quad n = \sum_{i\in I} n_i, \quad r = \sum_{s\in J} r_s, \quad \mu = \sum_{i\in I} \mu_i,$$

$$\overline{Z}_i = \left\{z_j : \ j \in I_i\right\}, \qquad i \in I, \quad \overline{\mathcal{U}}_i = \left\{u_j : \ j \in J_i\right\}, \qquad i \in I,$$

$$Z = (z_1, \ldots, z_N) \in R^n, \quad \mathcal{U} = (u_1, \ldots, u_M) \in R^r, \quad P = (p_1, \ldots, p_N) \in R^\mu.$$

Let the state of the i-th subobject be determined by the following nonlinear dependence on the state of the subobjects influencing the state of the i-th subobject:

$$z_i = f_i\left(\overline{Z}_i, \overline{u}_i, p_i\right), \qquad i = 1, \ldots, N. \tag{1}$$

Here $f_i(.,.,.)$is a given n_i-dimensional function, continuously differentiable with respect to all its arguments, $i = 1,..,N$.

Sets of permissible values of internal parameters $\mathbb{P} \in R^\mu$and state values $\mathcal{U} \in R^r$ of control subobjects are specified, determined, for example, by positional restrictions:

$$\mathbb{P} : \underline{p}_i \le p_i \le \overline{p}_i, \quad i = 1, \ldots, N, \quad U : \underline{u}_s \le u_s \le \overline{u}_s, \quad s = 1, \ldots, M. \tag{2}$$

We will assume that the system of n relations (equations) (1) for all permissible values of internal parameters $\mathbb{P}$ and states $\mathcal{U}$ of the control object has a unique solution $Z \in R^n$.

The problem under consideration is the following. It is required to determine such permissible values of internal parameters of subobjects P and states of control objects $\mathcal{U}$ that deliver the minimum value of a given convex objective function continuously differentiable with respect to all its arguments

$$f_0 \left(Z \left(\mathcal{U}, P \right), \mathcal{U}, P \right) \to \min. \tag{3}$$

In (3) under $Z \left(\mathcal{U}, P \right) = \left(z_1 \left(\mathcal{U}, P \right), \ldots, z_n \left(\mathcal{U}, P \right) \right)$ we mean the solution of the system of relations (1) for given permissible values of the internal parameters $P \in \mathbb{P} \subset R^{\mu}$ of the states of control subobjects $\mathcal{U} \in U \subset R^r$.

To numerically solve the problem (1)-(3), we use first-order finite-dimensional conditional optimization methods.

For this, we introduce index sets $\tilde{I}_i \subset I$ and $\tilde{J}_s \subset J$ such that

$$\tilde{I}_i = \left\{ j : i \in I_j \right\}, \quad i \in I, \quad \tilde{J}_s = \left\{ j : s \in I_j \right\}, \quad s \in J. \tag{4}$$

The set $\tilde{I}_i$ includes indices of those subobjects whose state is affected by the i-th subobject. Thus, if $j \in \tilde{I}_i$ and $s \in J_i$, then in (1) there is a dependencies

$$z_j = f_j \left(\ldots, z_i, \ldots \right), \quad z_i = f_i \left(\ldots, u_s, \ldots \right).$$

From (3), (4) it is easy to complete the lemma.

Lemma 1. From definitions (1), (2), (3), (4) it follows that the following relations are valid for all $i, j \in I$:

$$\frac{\partial z_i}{\partial z_j} \equiv \begin{cases} O_{n_i \times n_j}, \ j \in I_i \ or \ i \in \tilde{I}_j, \\ O_{n_i \times n_j}, \ j \notin I_i \ or \ i \notin \tilde{I}_j. \end{cases} \frac{\partial z_i}{\partial u_s} \equiv \begin{cases} O_{n_i \times r_i}, \quad s \in J_i \ or \ i \in \tilde{J}_i, \\ O_{n_i \times r_i}, \quad s \notin J_i \ or \ i \notin \tilde{J}_i. \end{cases} \tag{5}$$

In formulas (5), the derivatives on the left-hand sides mean the partial derivative with respect to the variables explicitly included in the right-hand sides of (1).

The following theorem holds, the proof of which follows from relations (1) and the definition of index sets I_i, $\tilde{I}_i$, J_j, $\tilde{J}_j$, $i \in I$, $j \in J$.

Theorem 1. The gradient components of the objective function (2) which is continuously differentiable with respect to all its arguments under conditions (1) are determined by the formulas:

$$\frac{df_0\left(Z\left(\mathcal{U},P\right),\mathcal{U},P\right)}{du_j} = \frac{\partial f^0\left(Z\left(\mathcal{U},P\right),\mathcal{U},P\right)}{\partial u_j} + \sum_{i\in\tilde{J}_j} \frac{\partial f_i\left(Z_i\left(\mathcal{U},P\right),\mathcal{U}_i,p_i\right)}{\partial u_j} w_i, j \in J,$$

$$\frac{df_0\left(Z\left(\mathcal{U},P\right),\mathcal{U},P\right)}{dp_i} = \frac{\partial f^0\left(Z\left(\mathcal{U},P\right),\mathcal{U},P\right)}{\partial p_i} + \frac{\partial f_i\left(Z_i\left(\mathcal{U},P\right),\mathcal{U}_i,p_i\right)}{\partial p_i} w_i, i \in I,$$

where the vectors $w_i \in R^{n_i}$, called conjugate, are the solution to the following system of equations:

$$w_i = \frac{\partial f_0\left(Z\left(\mathcal{U},P\right),\mathcal{U},P\right)}{\partial z_i} + \sum_{j\in\tilde{I}_i} \frac{\partial f_j\left(Z_j\left(\mathcal{U},P\right),\mathcal{U}_j,p_j\right)}{\partial z_i} w_j, i \in I$$

The paper studies a finite-dimensional optimization problem with special structure constraints. The Jacobian of equality type constraints has a large dimension, is sparse and has an arbitrary filling. The obtained formulas make it possible to use known effective first-order optimization methods to solve the problem.

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Application of mathematical modeling methods to describe the behavior of sandwich panels with a tetrahedral core

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Sandwich panels with tetrahedral core structure were considered under bending load and were represented as homogeneous panels [1] with anisotropic mechanical properties. Effective physical and mechanical characteristics were determined, representing the body as a homogeneous anisotropic continuum [2]. The results analysis shows that to correctly describe the stress-strain state of this type of panel at the macro level, in case different elastic behavior occurs under tension and compression, we need to employ the bi-modular theory of elasticity[3].

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The Cauchy problem for a double nonlinear parabolic equation with a critical exponent

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We consider the self-similar solution connected to the following nonlinear problem with a source under the Cauchy condition:

$$|x|^{-l_1}\partial_t u = u^q div\left(|x|^n\left|\nabla u^k\right|^{p-2}\nabla u^m\right)+|x|^{-l_2}t^l u^{\beta_c}, (x,t)\in D=R^N\times D_t \quad (1)$$

$$u|_{t=t_0}=M\delta(x), x\in R^N, \quad (2)$$

where $q<1, m,k\geq 1, p>2, l_1<N, l\geq 0, l_2, n=const$ are given parameters, $\beta_c=1+(l+1)\left[m+k(p-2)+2q-1+\frac{(1-q)(p-n-l_1)}{N-l_1}\right]+\frac{(1-q)(l_2-l_1)}{N-l_1}>1.$ - the critical Fujita exponent [3], $D_t=\{t|\, t>t_0=const>0\}$, $\delta(x)$ - Dirac delta function and $M=\int_{R^N}|x|^{-l_1}u(t,x)\,dx$.

Equation (1) characterizes many processes such as diffusion [3], heat dissipation [4], salt or dust transfer, biological population [2], image processing (Peron-Malik equation) and other processes. The equation (1) is studied by many authors.

The authors [5], studied problem (1)-(2) in the case $q=l_2=0$, $k=1$ and $q=l_1=l_2=0$, $k=1$. They establish the conditions for the existence and non-existence of global solutions of the Cauchy problem.

Furthermore, recall some well-known results. In particular, when $l_2=l_1$ authors of the work [1], studied the heat conduction equation with nonlinear source term. They showed that for the Cauchy problem, the critical Fujita exponent is: $\beta_c=1+(l+1)(m+k(p-2)+q-1)+\frac{(l+1)(p-l_1-n)}{N-l_1}$.

It is well-known that degenerate equations need not possess classical solutions. Therefore, in this case, we need to consider a weak solution from having a physical sense class.

We introduce the notation $v=u^{1-q}$ and put it into the problem (1)-(2). Then, we rewrite the problem (1)-(2) as follows

$$r^{-l_1}\frac{\partial_t v}{1-q}=r^{1-N}\frac{\partial}{\partial r}\left(r^{N-1+n}\left|\frac{\partial v^{k_2}}{\partial r}\right|^{p-2}\frac{\partial v^{m_2}}{\partial r}\right)+r^{-l_2}t^l v^{\beta_{2c}}, \qquad (x,t)\in D \quad (3)$$

$$v|_{t=t_0} = v_0(x) = [u_0(x)]^{1-q}, x \in R^N. \tag{4}$$

Where $m_2 = \frac{m}{1-q}, k_2 = \frac{k}{1-q}, \beta_{2c} = \frac{\beta_c - q}{1-q}, r = ||x|| = \sqrt{\sum_{i=1}^{N} x_i^2}$.

We look for the solution of (3) in the following form

$$v(r,t) = \bar{u}(t) w\left(\psi(t), r\right)$$

where $\psi\left(t\right) = \int \left[\bar{u}(t)\right]^{-\frac{m_2 + k_2(p-2)-1}{\beta_{2c}-1}} dt \approx \tau^{\alpha-1}\left(t\right) e^{(l+1)(\alpha-1)\tau(t)}$, $\tau\left(t\right) = \ln t$, $\bar{u}(t) = \left[t^{l+1} \ln t\right]^{-\frac{1}{\beta_{2c}-1}}$, $\alpha = \frac{\beta_{2c} - m_2 - k_2(p-2)}{\beta_{2c}-1}$.

It is easy to check that for an unknown function w obtained the following equation

$$\frac{r^{-l_1}}{1-q}\frac{\partial w}{\partial \psi} = r^{1-N}\frac{\partial}{\partial r}\left(r^{n+N-1}\left|\frac{\partial w^{k_2}}{\partial r}\right|^{p-2}\frac{\partial w^{m_2}}{\partial r}\right) +$$

$$\frac{r^{-l_1}}{1-q}\frac{1+(l+1)\tau}{(\beta_{2c}-1)e^{\tau}\psi} w + \frac{r^{-l_2}}{e^{\tau}\psi} w^{\beta_{2c}}. \tag{5}$$

Below, a method of nonlinear splitting [4] is provided to construct a self-similar equation in the following form:

$$w(\psi, r) = f\left(\xi\right), \xi = r\psi^{-\frac{1}{p-n-l_1}}. \tag{6}$$

Then, we obtain the following equation:

$$\xi^{1-N-l_1}\frac{d}{d\xi}\left(\xi^{n+N-1}\left|\frac{df^{k_2}}{d\xi}\right|^{p-2}\frac{df^{m_2}}{d\xi}\right) +$$

$$\frac{\xi}{(1-q)(p-n-l_1)}\frac{df}{d\xi} + \frac{1+(l+1)\tau}{(1-q)(\beta_{2c}-1)e^{\tau}} f + \frac{\xi^{l_1-l_2}}{e^{\tau}}\psi^{\frac{l_1-l_2}{p-n-l_1}} f^{\beta_{2c}} = 0. \tag{7}$$

We take the following auxiliary function

$$f\left(\xi\right) = A\left(a - \xi^{\gamma_1}\right)_+^{\gamma_2}, \tag{8}$$

where $\gamma_1 = \frac{p-n-l_1}{p-1}$, $\gamma_2 = \frac{p-1}{m_2+k_2(p-2)-1}$, $A = \left[\gamma_1\gamma_2 m_2\left(1-q\right)\left(p-n-l_1\right)k_2^{p-2}\right]^{-\gamma_2}$, $(d)_+ = max\left(d, 0\right), a = const \geq 0$.

Let us denote $z\left(x,t\right) = \bar{u}(t) f\left(\xi\right)$.

Theorem 1. *Let* $\gamma_2 > 0, p > n + l_1, v(x,0) \leq z(x,0), x \in R^N$ *and the following inequality be satisfied:*

$$t \geq \max\left\{\exp\left(-W_0\left(-n_2/\left(l+1\right)\cdot\exp\left(-n_1/\left(l+1\right)\right)\right)-n_1/\left(l+1\right)\right), t_0\right\}$$

Then for the solution to the problem (3)-(4) an estimate

$$v\left(x,t\right) \leq z\left(x,t\right) \quad \text{in D}$$

hold.

Where $n_1 = p - n - l_1 > 0$, $n_2 = (N + l_1)(\beta_{2c} - 1) > 0$, and W_0- the Lambert function.

Proof. The proof of the Theorem 1 is similar to the proof of theorems in [1].

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Numerical modeling of diffusion processes in two-component nonlinear media with variable density and source

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In this work in the domain $Q = \left\{(t,x): \ 0<t, \ \ x \in R^N\right\}$ consider a reaction-diffusion system with a double nonlinearity and a variable density in a two-componential medium described by the following system of degenerate parabolic equations with variable density and time-dependent nonlinear source or absorption

$$\begin{array}{l} |x|^{-l}\frac{\partial u}{\partial t} = \nabla\left(|x|^n u^{m_1-1}\left|\nabla u^k\right|^{p-2}\nabla u^{l_1}\right) + \varepsilon|x|^{-l}\gamma(t)u^{p_1}v^{q_1} = 0, \\ |x|^{-l}\frac{\partial v}{\partial t} = \nabla\left(|x|^n v^{m_2-1}\left|\nabla v^k\right|^{p-2}\nabla v^{l_2}\right) + \varepsilon|x|^{-l}\gamma(t)u^{p_2}v^{q_2} = 0, \end{array} \quad (1)$$

with initial condition

$$u(0,x) = u_0(x) \geq 0, \ \ v(0,x) = v_0(x) \geq 0, \quad x \in R^N, \quad (2)$$

where, $k \in R, \ m_1, m_2 > 1, \ p_i, q_i \geq 1, \ p \geq 2$ are positive real numbers, and, $u_0(x) \geq 0, v_0(x) \geq 0$- are a non-trivial, non-negative, bounded and sufficiently smooth functions, $0 < \gamma(t) \in C(0,\infty)$.

System (1) describes various physical processes in a two-component nonlinear medium of the reaction-diffusion, heat conduction, combustion, polytropic filtration of liquid and gas processes with variable density at the presence of a source or an absorption power of which is equal to $\gamma(t)u^{p_i}v^{q_i}$.

Problems (1)-(2) studies by many authors for the particular value of the numerical parameters have been studied intensively by numerous authors (see [1-6] and references therein). For the investigation qualitative properties of the considered problem, we suggest the method of nonlinear splitting for construction of an approximately self–similar approach. To reduce the system of equations (1) to an approximately self-similar we consider

$$\begin{array}{l} u\left(t,x\right) = \overline{u}\left(t\right) w\left(\tau\left(t\right), \varphi\left(|x|\right)\right), \ \ v\left(t,x\right) = \overline{v}\left(t\right) z\left(\tau\left(t\right), \varphi\left(|x|\right)\right), \\ w\left(\tau\left(t\right), \varphi\left(x\right)\right) = f_1\left(\xi\right), \ \ \ z\left(\tau\left(t\right), \psi(x)\right) = f_2\left(\xi\right), \end{array} \quad (3)$$

where

$$\overline{u}\,(t) = \left(T + \int\limits_0^t \gamma\,(y)\,dy\right)^{-\alpha_1}, \qquad \overline{v}\,(t) = \left(T + \int\limits_0^t \gamma\,(y)\,dy\right)^{-\alpha_2},$$

are solutions of the system of ordinary differential equations

$$\frac{d\overline{u}}{dt} = -\gamma(t)\overline{u}^{p_1}\overline{v}^{q_1}, \quad \frac{d\overline{v}}{dt} = -\gamma(t)\overline{u}^{p_2}\overline{v}^{q_2} \tag{4}$$

where

$$\alpha_1 = \frac{(q_2+1)-q_1}{(p_1-1)(q_2-1)-p_2q_1}, \quad \alpha_2 = \frac{(p_1+1)-p_2}{(p_1-1)(q_2-1)-p_2q_1},$$

$$\tau\,(t) = \int\limits_0^t u^{k(p-2)+m_1+l_1-2}\,(y)\,dy = \int\limits_0^t v^{k(p-2)+m_2+l_2-2}\,(y)\,dy$$

Substituting (3) to the system (1) it reduced to the following system of equations

$$\begin{aligned} \frac{\partial w}{\partial \tau} &= \varphi^{1-s}\frac{\partial}{\partial \varphi}\left(\varphi^{s-1}w^{m_1-1}\left|\frac{\partial w}{\partial \varphi}^k\right|^{p-2}\frac{\partial w}{\partial \varphi}^{l_1}\right) + \bar{u}^{p_1-k(p-2)-m_1-l+1}\bar{v}^{q_1}\left(w+\varepsilon w^{p_1}z^{q_1}\right) \\ \frac{\partial z}{\partial \tau} &= \varphi^{1-s}\frac{\partial}{\partial r}\left(\varphi^{s-1}z^{m_2-1}\left|\frac{\partial z}{\partial \varphi}^k\right|^{p-2}\frac{\partial z}{\partial \varphi}^{l_2}\right) + \bar{u}^{p_2}\bar{v}^{q_2-k(p-2)-m_2-l+1}\left(z+\varepsilon z^{q_2}w^{p_2}\right) \end{aligned} \tag{4}$$

where $s = \frac{p(N-l)}{p-(n+l)}$, $n+l<p$

It is easy to see that system (5) has the following an approximately self-similar solution of the form

$$w\,(\tau\,(t)\,,\varphi\,(x)) = f_1\,(\xi)\,, \quad z\,(\tau\,(t)\,,\psi(x)) = f_2\,(\xi)\,, \tag{5}$$

where $\varphi\,(x) = \frac{p}{p-(n+l)}|x|^{\frac{p-(n+l)}{p}}$, $\xi = \varphi\,(x)\,\tau^{-1/p}$ and the functions $f_1\,(\xi)$, $f_2\,(\xi)$ satisfy to the approximately self-similar system in particular if $\gamma(t)=1$ then

$$\begin{aligned} &L_1(f_1,f_2) \equiv \xi^{1-s}\frac{d}{d\xi}\left(\xi^{s-1}{f_1}^{m_1-1}\left|\frac{df_1}{d\xi}^k\right|^{p-2}\frac{df_1}{d\xi}^{l_1}\right) + \frac{\xi}{p}\frac{df_1}{d\xi} + \\ &\frac{\alpha_1}{1-(k(p-2)+m_1+l_1-1)\alpha_1}\left(f_1+\varepsilon {f_1}^{p_1}{f_2}^{q_1}\right) = 0, \\ &L_2(f_1,f_2) \equiv \xi^{1-s}\frac{d}{d\xi}\left(\xi^{s-1}{f_2}^{m_2-1}\left|\frac{df_2}{d\xi}^k\right|^{p-2}\frac{d{f_2}^{l_2}}{d\xi}\right) + \frac{\xi}{p}\frac{df_2}{d\xi} + \\ &\frac{\alpha_2}{1-(k(p-2)+m_2+l_2-1)\alpha_2}\left(f_2+\varepsilon {f_2}^{q_2}{f_1}^{p_2}\right) = 0. \end{aligned} \tag{6}$$

$$(k(p-2)+m_1+l_1-1)\alpha_1 = (k(p-2)+m_2+l_2-1)\alpha_2$$

where

$$\begin{aligned} &f_1(\xi) = \overline{f_1}(\xi)\, w(\tau)\,, \ \ \overline{f_1}(\xi) = (a-\xi^\gamma)_+{}^{n_1},\ \gamma = \tfrac{p}{p-1},\ \ \tau = -\ln(a-\xi^\gamma), \\ &f_2(\xi) = \overline{f_2}(\xi) z(\tau),\ \ \overline{f_2}(\xi) = (a-\xi^\gamma)_+{}^{n_2},\ \ \xi = \varphi(x)\,\tau(t)^{-1/p}, \end{aligned} \tag{7}$$

Consider self-similar solutions to system (7) satisfying the following boundary conditions:

$$\begin{aligned} f_1(0) = c_0 > 0,\ \ f_1(b_1) = 0,\ \ b_1 < \infty, \\ f_2(0) = c_0 > 0,\ \ f_2(b_2) = 0,\ \ b_2 < \infty, \end{aligned} \tag{8}$$

$$f_1(0) = c_0 > 0,\ \ f_1(\infty) = 0,\ f_2(0) = c_0 > 0,\ \ f_2(\infty) = 0. \tag{9}$$

Theorem 1. Let $m_i + l_i + k(p-2) - 1 > 0, \ \ i = 1,2.$ Then the solution $f_1(\xi),\ f_2(\xi)$ of the system (7) for $\xi \to a^{\frac{1}{\gamma}}$ has asymptotics

$$\begin{aligned} f_1(\xi) = c_1 \bar{f}_1(\xi)\,(1+o(1)) \\ f_2(\xi) = c_2 \bar{f}_2(\xi)\,(1+o(1)) \end{aligned} \tag{10}$$

where coefficients c_1, c_2 satisfy the system of algebraic equations

$$\begin{aligned} (n_1\,(k(p-2)+m_1+l_1)-p+1)\,\gamma n_1 |kn_1|^{p-2} c_1{}^{m_1+k(p-2)+l_1} + a c_1{}^{p_1} c_2{}^{q_1} = 0, \\ (n_2\,(k(p-2)+m_2+l_2)-p+1)\,\gamma n_2 |kn_2|^{p-2} c_2{}^{m_2+k(p-2)+l_2} + a c_1{}^{p_2} c_2{}^{q_2} = 0. \end{aligned} \tag{11}$$

Theorem 2. Let $m_i + l_i + k(p-2) - 1 < 0, \ \ i = 1,2.$ Then the regular solution $f_1(\xi),\ f_2(\xi)$ of the problem (7)- (10) for $\xi \to \infty$ has asymptotics

$$f_1(\xi) = c_1 \bar{f}_1(\xi)(1+o(1)),\ f_1(\xi) = c_2 \bar{f}_2(\xi)(1+o(1)) \tag{12}$$

where the coefficients c_1, c_2satisfy the system of algebraic equations

$$\begin{aligned} (s + (n_1(k(p-2)+m_1)-p+1))\,\gamma n_1 |kn_1|^{p-2} c_1{}^{m_1+k(p-2)} + c_1{}^{p_1} c_2{}^{q_1} = 0, \\ (s + (n_2(k(p-2)+m_2)-p+1))\,\gamma n_2 |kn_2|^{p-2} c_2{}^{m_2+k(p-2)} + c_1{}^{p_2} c_2{}^{q_2} = 0 \end{aligned} \tag{13}$$

It is investigated properties of solution in the critical $(m_i + k(p-2) - 1 = 0)$ and double critical and singular cases $(p = n + l)$

In this paper, the qualitative properties of the problem Cauchy for a double nonlinear system with variable density and nonlinear source and absorption based on self-similar analysis of solutions, the influence of variable density, source or absorption to the evolution of studied processes established. The asymptotic behavior of self-similar solutions depends on the value of the numerical parameters of the system (1) studied. The problem of choosing initial approximations for the numerical analysis of solutions of the considered problem is solved. It is shown that the coefficient at the principal term of the asymptotics of the solution satisfies some system of nonlinear algebraic equations. For numerical solutions, the iterative processes are built on the basis of the Picard, Newton methods. The results of computational experiments show that both iterative methods are effective for numerical solutions considered double nonlinear problems and lead to new nonlinear effects due to the suggested initial approximation the solutions for the numerical solution.

The results of computational experiments show that both the iterative method and computational scheme and sweep iterative methods are effective for solving considered nonlinear problems and lead to nonlinear effects if solutions of self-similar equations constructed by the nonlinear splitting and the standard equation methods are used as the initial approximation.

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A counterexample to a conjecture on semiclassical orthogonal polynomials

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In an earlier work [K. Castillo et al., J. Math. Anal. Appl. 514 (2022) 126358], we give a positive answer to the first, and apparently more easy, part of a conjecture of M. Ismail concerning the characterization of the continuous q- Jacobi polynomials, Al-Salam-Chihara polynomials, or special or limiting cases of them. In this note, we present an example that disproves the second part of such a conjecture, and so this issue is definitively closed.

The practice of finite element modeling in civil engineering structures

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Situations in which soil masses are subject to large displacement are ubiquitous in geotechnical and civil engineering structures: sampling, in situ testing, pile installation, landslides, retaining walls, to name a few. Numerical simulation is clearly useful to advance understanding in these areas. The phenomenon of slope instability is a problem of interest for researchers and patricians in the field of geotechnical engineering. However, the analytical and numerical simulation of such problems is a complex task, since the system is full of nonlinearities, material-related, and also geometrical. The finite elements method (FEM), which has proven successful for the numerical simulation of several problems, becomes unreliable once geometric nonlinearities. Therefore, the purpose of this paper is to make an analytical and numerical study of the stability of a reference slope located in Kherrata City (Bejaia Governorate, Northern Algeria) (case study). Limit equilibrium and finite element methods were using in the analysis. The results obtained will be the subject of a comparative study between the calculations of the safety factor by the classical methods and the finite elements method integrated in the PLAXIS 2D software. The main findings indicate that both limit equilibrium methods and FEM were converging.

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Multicontinuum homogenization and multiscale finite element methods

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Abstract. Many problems have multiscale nature. For example, the flow in porous media occurs in multiscale media with heterogeneities at multiple scales and high contrast. The simulations of these problems are often performed on a coarse computational grid, where the grid size is much larger compared to the scales of heterogeneities. In these simulations, we distinguish two cases. The first is the case with no-scale separation and the second is the case with scale separation. In the first case, approaches use the information within the entire computational grid or beyond to derive macroscopic equations. In the second case, the representative volume-based information (which is much smaller compared to the target coarse block) is used in deriving macroscopic equations.

In this talk, we discuss a general framework for multicontinuum homogenization. Multicontinuum models are widely used in many applications and some derivations for these models are established. In these models, several macroscopic variables at each macroscale point are defined and the resulting multicontinuum equations are formulated. In this talk, we propose a general formulation and associated ingredients that allow performing multicontinuum homogenization. Our derivation consists of several main parts. In the first part, we propose a general expansion, where the solution is expressed via the product of multiple macro variables and associated cell problems. The second part consists of formulating the cell problems. The cell problems are formulated as saddle point problems with constraints for each continua. Defining the continua via test functions, we set the constraints as an integral representation. Finally, substituting the expansion to the original system, we obtain multicontinuum systems. We present an application to the mixed formulation of elliptic equations. This is a challenging system as the system does not have symmetry. We discuss the local problems and various macroscale representations for the solution and its gradient. Using various order approximations, one can obtain different systems of equations. We discuss the applicability

of multicontinuum homogenization and relate this to high contrast in the cell problem.

We introduce a general homogenization method, where we assume that the media properties can have high contrast. In our expansion, we consider that each macroscopic point has several macroscopic variables associated with it. The macroscopic variables are defined via auxiliary functions and assumed to be smooth functions. The expansion of the solution via macroscopic variables uses the solution of local microscopic problems posed in RVE, called solutions of cell problems. These local problems account for the micro-scale behavior of the solution given certain constraints. These constraints are related to the definition of macroscopic variables. In particular, our first cell problem imposes constraints to represent the constants in the average behavior of each continua. The consequent cell problems impose constraints to represent the high-order polynomials in the average behavior of each continua.

One of the challenging aspects of multicontinuum homogenization is in formulating cell problems correctly. We consider large oversampled regions, where we can impose higher-order polynomial constraints. By imposing averages in each RVE within the oversampled region, our main test is to guarantee that the solution of the cell problem converges to zero. To achieve this, one needs a careful formulation of cell problems. For example, in a carefully studied example of mixed Darcy equations, we show how one can achieve this. We obtain a generalized Darcy approximation on the coarse grid. We present an error analysis for our multicontinuum approach.

We present numerical examples. In this numerical example, we consider a mixed formulation between velocity and pressure in Darcy's equation. Because pressure and velocity are treated separately, their relation at the microscale will not necessarily preserve at the macroscale as in the standard homogenization. We note that there is a linear relation between the velocity and the gradient of the pressure via the multiscale permeability field. Because the mixed formulation is not symmetric, this causes further challenges that are addressed in numerical examples when imposing local constraint problems. Our numerical results show a good convergence as we decrease the mesh size. field. Because the mixed formulation is not symmetric, this causes further challenges that are addressed in numerical examples when imposing local constraint problems. Our numerical results show a good convergence as we decrease the mesh size.

Co-existence of 4 types of synchronization in 4-D hyperchaotic dynamical systems

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Chaos has become a focal point for nonlinear problems in subjects ranging from physics and chemistry to mechanics and communications. Synchronization of chaotic systems have several important applications in science and engineering such as oscillators, lasers, cryptosystems, secure communications, biology, robotics, networks, transmission, electronics, mechanics, etc. Many different types of synchronization phenomena have been intensively investigated and a lot of theoretical results have been obtained in the past 20 years, such as complete synchronization (CS), projective synchronization (PS), hybrid projective synchronization (HPS), full state hybrid projective synchronization (FSHPS), etc, using different types of control scheme such as linear and non-linear feedback synchronization, adaptive control, active control, sliding mode control, backstepping control, etc.In this paper, based on the Lyapunov stability theory, different schemes of synchronization for hyperchaotic dynamical systems, such CS, PS, HPS, and FSHPS, are combined to derive and achieve synchronization of two coupled hyperchaotic dynamical systems in 4-D continuous-time. Numerical simulations carried out to validate the theoretical result and verify the effectiveness of the proposed scheme of synchronization.

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Generalized Sombor index of some well known families of graphs

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A topological index is a function $TOP : \mathcal{G} \to \mathcal{R}$, whose domain is the set of graph families and its co-domain is the set of positive real numbers. For a (molecular) graph, the Somber, Reduced Sombor and Generalized Sombor indices are respectively denoted by $SO(G)$, $SO_{red}(G)$ and $GSO(G)$ and defined as:

$$SO(G) = \sum_{uv \in E(G)} \sqrt{((d(u))^2 + (d(v))^2)}$$

$$SO_{red}(G) = \sum_{uv \in E(G)} \sqrt{((d(u) - 1)^2 + (d(v) - 1)^2)}$$

$$GSO(G) = \sum_{uv \in E(G)} \left((d(u))^2 + (d(v))^2\right)^\gamma$$

Where $\gamma \geq \frac{1}{2}$ is a real number, and $d(u), d(v)$ represents degrees of the vertex u and v respectively. In this paper we present results, how to obtain optimal graph from the family of trees, unicyclic graphs and bicyclic graphs whose $SO(G), SO_red(G)$ and $GSO(G)$ parameters are either maximum or minimum among all graphs present in that family. We claim to obtain these results by using some transformations, that we are going to introduce in this article.

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A Numerical scheme for solving a class of variable order differential equations

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Fractional calculus, which denotes differentiation and integration with fractional orders, has a history spanning more than three centuries. Many real-world phenomena can be more accurately described using fractional operators rather than ordinary calculus [1].

In 1993, Ross and Samko presented the concept of variable-order (VO) fractional operators and evaluated their features [2]. To precisely describe complicated physical systems and procedures, VO fractional calculus (VO-FC) has attracted attentions as a mathematical structure [3]. Since then, numerous researchers have generalized the VO fractional calculus, and its applications have been researched in numerous fields, such as petroleum engineering, viscoelasticity oscillators, engineering, signal processing, etc [4].

Because the VO operators have a variable exponent kernel, the analysis of variable-order systems is more complicated than that of constant-order systems [5]. Obtaining analytical solutions is frequently challenging or even impossible. Therefore, numerous scholars are investigating numerical methods[4].

In this work, we use the operational matrices (OMs) and collocation method (CM) to obtain numerical solution for a class of variable order differential equations(VO-DEs) in the following form:

$$\begin{cases} {}^{C}\mathcal{D}_t^{\beta_1(t)}x(t) = f(t, x(t), {}^{C}\mathcal{D}_t^{\beta_2(t)}x(t), \cdots, {}^{C}\mathcal{D}_t^{\beta_n(t)}x(t)), \quad \underbrace{k_i - 1 < \beta_i(t) \leq k_i}_{\{k_i \in \mathbb{Z}^+\}_{i=1}^{n\in\mathbb{N}}}, \quad 0 \leq t \leq 1, \\ x^{(i)}(0) = a_i \in \mathbb{R}, \qquad i = 0, 1, \cdots, k_1 - 1, \end{cases} \tag{6}$$

where $x(t)$ is the unknown function and ${}^{C}\mathcal{D}_t^{\beta_i(t)}$, with condition $\beta_1(t) > \beta_2(t) > \cdots > \beta_n(t)$ is the Caputo derivative of VOs, which is defined as follows [6]

$${}^{C}\mathcal{D}_t^{\beta_i(t)}x(t) = \frac{1}{\Gamma(k_i - \beta_i(t))} \int_0^t (t-s)^{k_i - \beta_i(t) - 1} x^{(k_i)}(s)\, ds.$$

The aim of this study is to convert problem (6) into an algebraic system of equations. For this purpose, we use operational matrices and collocation method. In other words, (6) is transformed into several dependent matrices with the help of the operational matrices, which is converted to an algebraic system of equation by using the collocation points.

The fractional derivatives and the VO-derivatives are in the Caputo sense. The operational matrices are computed based on the Hosoya polynomials (HPs) of simple paths. Firstly, we assume the unknown function as a finite series by using the Hosoya polynomials as the basis functions. To obtain unknown coefficients of this approximation, we computed the operational matrices of all terms of the main equations. Then, by using the operational matrix and collocation points, the governing equations converted to a set of algebraic equations. Finally, An approximate solution is obtained by solving that algebraic equations.

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Symmetry transformations of parabolic equations

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In this talk, we discuss the one-parameter Lie group transformations that leave differential equations invariant. In particular, we consider parabolic models that have a long history of interesting properties. We link transformations to a class of these equations and showcase how certain simpler models may be found that are easily solvable subject to initial conditions. As an example, we discuss a portfolio optimization problem.

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Numerical study of transonic flow in a channel at various back pressures

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Studies of high-speed airflow in convergent-divergent or curved channels are of practical interest, e.g., for the advanced design of supersonic aircraft engines. The efficiency of the engine depends crucially on the intake that decelerates the incoming atmospheric airflow to velocities suitable for fuel burning in the combustor. In on-design conditions, the deceleration of supersonic flow is accomplished through several oblique shock waves located inside the intake. Such a flow regime is known as started intake [1]. Meanwhile, if the inflow Mach number M_∞ decreases or the pressure imposed at the exit p_{back} increases, then the shock waves shift upstream and eventually are expelled from the intake. The latter regime, which is called an unstarted intake, produces considerable losses in the engine thrust.

A number of works addressed the flow behavior in convergent-divergent intakes and bent channels under changes in freestream Mach number M_∞ or angle of attack [2]. Also, a point of importance is the flow behavior under changes in the back pressure p_{back}. Zhao *et al.* [3] carried out an experimental study of the effect of back pressure on the flow in a channel whose lower wall involves a bend of 8°; a noticeable hysteresis was detected during the unstart/restart of the intake.

In this paper, we consider the same channel as in [3] and perform a numerical study of the flow. The flow is governed by the system of Reynolds-averaged Navier-Stokes differential equations [4] with respect to static pressure $p(x, y, t)$, temperature $T(x, y, t)$, and velocity components $U(x, y, t), V(x, y, t)$, where (x, y) are the Cartesian coordinates and t is time.

Figure 1 illustrates the geometry of the channel and outer boundaries of the computational domain. Details of the geometry are available in [3]. On the inflow boundary of the computational domain, we impose velocity components $U_\infty = 592.08$, $V_\infty = 83.21$ m/s, temperature $T_\infty = 122.1$ K, and pressure $p_\infty = 4,300$ Pa, which correspond to $M_\infty = 2.7$, as in [3]. At the exit of the channel, we prescribe the pressure p_{back}. On the walls of the channel, the flow velocity vanishes. Initial conditions are either the uniform freestream or non-uniform flow field obtained for another value of p_{back}. The stated initial-

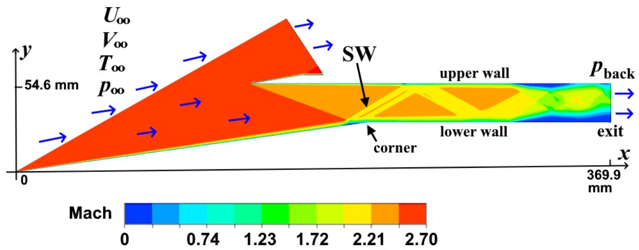

Figure 2: Mach number contours in the channel at $M_\infty = 2.7$, p_{back}=24,000 Pa

boundary value problem was solved numerically with an ANSYS CFX finite-volume solver on an unstructured mesh constituted by 948,119 cells. At steady values of p_{back}, transient solutions of the problem showed a fast convergence to steady states. The obtained steady solutions $p(x, y), T(x, y), U(x, y), V(x, y)$ made it possible to identify locations of shock waves in the channel, as well as unstart/restart conditions.

In particular, the solutions revealed a shock wave SW induced by the boundary-layer separation ahead of the lower wall corner, see Fig. 1. Changes in the pressure p_{back} showed that the shock SW persists when p_{back} increases step-by-step from 1,000 Pa to 33,600 Pa, whereas the system of oblique shocks gradually shifts upstream in the region between the corner and exit. If p_{back} is further increased to 33,800 Pa, then the shock SW jumps upstream to the entrance of the channel; this means a transition to the regime of unstarted intake. After that, a step-by-step decrease in p_{back} from 33,800 Pa to 32,100 Pa results in a gradual swallowing of SW and transition to the restarted intake. Therefore, there exists hysteresis in the band 32,100 Pa $\leq p_{\text{back}} \leq$ 33,800 Pa. Further decrease in p_{back} does not change the location of SW, though shifts the oblique shocks downstream of the bend towards the exit.

The hysteresis and transitions between different flow regimes are caused by the instability of the boundary-layer separation from the lower wall in contrast to the case of smaller freestream Mach number M_∞ [2] when hysteresis is caused by instability of the shock wave interaction with the flow expansion region over the corner.

We notice that, if p_{back} rises to 36,000 Pa, then the shock SW jumps upstream at a larger distance from the corner than that for 33,800 Pa; therefore, SW becomes expelled from the channel. In this case, numerical solutions show that any subsequent decrease in p_{back} cannot produce the swallowing of SW, i.e., do not provide a restart.

The results of the numerical study are in good agreement with experimental data documented in [3]. Minor discrepancies in oblique shock locations in the region between the corner and exit of the channel at decreasing p_{back} are explained by different ways of imposing the back pressure, since in [3] a transverse jet was used to rise the pressure near the exit, whereas in our simulations the average pressure over the exit section is set directly.

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Utilizing inverse matrices and determinants in analyzing the Zhegalkin polynomial

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There are several methods available for finding the Zhegalkin polynomial, such as equivalent transformations, Pascal's triangle, the method of uncertain coefficients, among others [1;2]. This article demonstrates that the Zhegalkin's polynomial can be expressed using determinants. Similarly, expressions can be formulated for other functions [5].

For two local functions $f(x,y)=(f_1,f_2,f_3,f_4)$ under the condition $f(x,y)=1$, let us construct the matrix D(x,y) and denote its inverse matrix by D^{-1}:

$$D(x,y)=\begin{bmatrix}0&1&x&y&xy\\ f_1&1&0&0&0\\ f_2&1&0&1&0\\ f_3&1&1&0&0\\ f_4&1&1&1&1\end{bmatrix};\ D^{-1}=\begin{bmatrix}e_0&e_{01}&e_{02}&e_{03}&e_{04}\\ c_0^{'}&c_{01}&c_{02}&c_{03}&c_{04}\\ c_1^{'}&c_{11}&c_{12}&c_{13}&c_{14}\\ c_2^{'}&c_{21}&c_{22}&c_{23}&c_{24}\\ c_3^{'}&c_{31}&c_{32}&c_{33}&c_{34}.\end{bmatrix}. \quad (1)$$

Theorem: Two local logical functions $f(x,y)=(f_1,f_2,f_3,f_4)$ can be written as follows:

$$f(x,y)=\frac{-1}{A_{11}}DetD(x,y) \pmod 2, \quad (1^{'})$$

where

$$A_{11}=\begin{vmatrix}1&0&0&0\\ 1&0&1&0\\ 1&1&0&0\\ 1&1&1&1\end{vmatrix}=\begin{vmatrix}0&1&0\\ 1&0&0\\ 1&1&1\end{vmatrix}=-1. \quad (2)$$

Proof: Let's use the following proof: Let's check the condition $f(0,0)=f_1$:

$$f(0,0)=\frac{-1}{A_{11}}\begin{bmatrix}0&1&0&0&0\\ f_1&1&0&0&0\\ f_2&1&0&1&0\\ f_3&1&1&0&0\\ f_4&1&1&1&1\end{bmatrix}=\frac{-1}{A_{11}}\begin{bmatrix}-f_1&1&0&0&0\\ f_1&1&0&0&0\\ f_2&1&0&1&0\\ f_3&1&1&0&0\\ f_4&1&1&1&1\end{bmatrix}=f_1.$$

It can be proved similarly in other conditions. The theorem is proved.

$$f(x,y) = c_0 + c_1 x + c_2 y + c_3 xy \pmod 2 \tag{3}$$

in order to identify the coefficients in the separation, it is necessary to select values for both x and y that correspond to the condition $f(x,y) = 1$. Consequently, the matrix $D(x,y)$ possesses an inverse matrix. The inverse of the matrix (1) the coefficients $c_i \quad i = (\overline{0,3})$ in the expression can be found: $c_i = \frac{-c_i^{'}}{A_{11}}$.

Note 1: $c_i^{'} \quad i = (\overline{0,3})$ coefficients can be found by another method. If we take only one of the elements in the first row of matrix (1) as 1 and the others as 0, that is, by calculating the determinants of the matrix, we can find the separation coefficient of (3) using the formula $(1^{'})$.

Note 2: It is appropriate to use functions from the maths category of Excel for calculations.

Example: The following distinctions $f_1(x,y) = x \bigvee y$ and $f_2(x,y) = x \bigoplus y$ are obtained:

$$f_1(x,y) = 0 + x + y + xy \pmod 2; \quad f_2(x,y) = x + y \pmod 2.$$

Note 3: It is possible to derive analogous formula for the three-digit logic function $f(x,y,z)$.

Note 4: The formula (1') is written in a similar manner to the determinant expression of the Lagrangian interpolation polynomial [3].

Note 5: It is also possible to write expressions for derivatives of Bul functions using $(1^{'})$. In a similar manner, it is possible to write the interpolation polynomial of multivariable functions (their special derivatives), high-order derivatives of the Lagrange interpolation polynomial [4], the integral of the interpolation polynomial, and the formula for polynomials according to the k-module.

Conclusion: Analogous to expressing of the Lagrange interpolation polynomial with determinants [3], it is feasible to represent higher-order derivatives of the interpolation polynomial [4], the Zhegalkin polynomial for logical functions, the interpolation polynomial of multivariable functions (including their special derivatives), definite and indefinite integrals of the interpolation polynomial [5], and polynomial formula according to the k-module. By utilizing inverse matrices and determinants the coefficients of Zhegalkin's polynomial can be computed through two distinct methods. Special examples illustrating this are provided.

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A comprehensive comparison in nonsmooth optimization methods, and their recent advances

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This study aims to provide a brief overview of nonsmooth optimization methods and their important applications. The following solvers are used in the comparisons.

- Multiobjective proximal bundle method (**MPBNGC**) for nonconvex, nonsmooth (nondifferentiable) and generally constrained minimization is an implementation of the Proximal Bundle Method in [8, 7].
- Limited memory bundle method (**LMBM**) for general, possible nonconvex, nonsmooth (nondifferentiable) large-scale minimization is an implementation of the Limited memory bundle method in [6].
- Discrete gradient method (**DGM**)for derivative-free optimization is an implementation of the discrete gradient method in [2].
- Quasi-Secant Method (**QSM**) for nonsmooth possibly nonconvex minimization is an implementation of the quasi-secant method in [1].
- Truncated codifferential method (**TCM**) for nonsmooth convex optimization is an implementation of the codifferential method [3].
- Hybrid Algorithm for Non-Smooth Optimization (**HANSO**) is an implementation of BFGS and gradient sampling method in [4].
- Gradient-based Algorithm for Non-Smooth Optimization (**GRANSO**) is an implementation of modifed versions of the BFGS and the inexact weak Wolfe line search in [5].

As is known, with the subgradient concept developed in the early 60s, various methods began to be developed for nonsmooth optimization problems. These methods have become a powerful tool for solving many important real-world problems. In this study, these methods were compared using some test problems. This comparison reveals the differences between the methods, their advantages and disadvantages. Test problems include nonsmooth unconstrained convex and non-convex problems with a large number of variables, as well as problems with special structures. It especially includes piecewise linear and piecewise quadratic problems. The article concludes with a brief discussion of possible future directions.

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A combination of ant colony optimization and simulated annealing for solving the multiple traveling salesman problem

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The multiple traveling salesman problem (MTSP) is one of the most important combinational optimization problems that has nowadays received much attention because of its real application in industrial and service problems [1]. The aim of this paper is to introduce a hybrid two-phase algorithm called MASE for solving the MTSP which can be explained as the problem of designing collection of tours from one depot to a number of customers [2]. At the first stage, the MTSP is solved by the elite ant system, and at the second stage, the simulated Annealing (SA) is used for improving solutions. This process avoids the premature convergence and makes better solutions. Extensive computational tests on standard instances from the literature confirm the effectiveness of the presented approach.

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Assessment of physiological development stages of flowering of plants in the cultivated area using spectral analysis of images

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In the present study, the study of the physiological development stages of the plant, the investigation of the ratio of plant-soil areas in the area was carried out by analyzing the distribution functions of spectral brightness values in the area covered by the image. The assessment of the flowering period, which is the physiological development stage of the plant in cultivated fields, is considered based on spectral images. To do this, spectral data characterizing the flower is selected, and statistical and neural network methods are used to identify the flower.

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Assessment of vegetation cover of territories using various indices

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This article develops an algorithm for classifying soil and vegetation cover using various indices. The study of the physiological stages of plant development is carried out by analyzing the distribution functions of spectral brightness values in the area covered by the image. The stages of the processing process are described here and the results of the corresponding calculations are presented.

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DIFFERENTIAL EQUATIONS AND OPTIMAL CONTROL

A study on controllability of second order neutral differential equations

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Abstract. In this article, we are concerned with a control system governed by a second-order non-autonomous neutral differential equation in a reflexive Banach space X. We study the existence of a mild solution and the system's approximate controllability. To establish the results, we use the semigroup theory of bounded linear operators, resolvent operator theory, Arzela-Ascoli theorem, and Krasnoselskii's fixed point theorem. In the end, we demonstrate our abstract results by an example.

The mathematical model of the considered problem is as follows:

$$\begin{cases} \dfrac{d^2}{dt^2}N(t,x_t) = A(t)N(t,x_t) + \displaystyle\int_0^t \psi(t,s)N(s,x_s)ds + Bu(t) \\ \qquad\qquad\qquad\qquad +E_1(t,x_t), \quad t \in I = [0,T], \\ x_0(t) = \mu(t) \in P, \quad t \in I_0 = (-\infty, 0], \\ x'(0) = x_1 \in X, \end{cases} \tag{7}$$

where the state function $x(\cdot)$ takes values in $X, A(t) : D(A(t)) \subseteq X \to X$ and $\Psi(t,s) : D(\Psi) \subseteq X \to X$ are closed linear operators. We also consider that $D(\Psi)$ is independent of (t,s). The history function $x_t : (-\infty, 0] \to X, x_t(s) = x(t+s)$ belongs to some abstract phase space P (introduced by Hale and Kato [4]). Phase space $(P, \|\cdot\|)$ is a Banach space satisfying the following properties:

If $x : (-\infty, \nu + T) \to X, T > 0, \nu \in R$ is piecewise continuous on $[\nu, \nu + T)$ and $x_\nu \in P$, then for every $t \in [\nu, \nu + T)$, we have

(i) $x_t \in P$

(ii) $\|x(t)\| \leq K_1 \|x_t\|_P$

(iii) $\|x_t\| \le K_2(t-\nu)\sup\{\|x(s)\| : \nu \le s \le t\} + K_3(t-\nu)\|x_\nu\|$, where $K_1 > 0$ is a constant, $K_2, K_3 : [0,\infty) \to [0,\infty)$, K_2 is continuous and K_3 is locally bounded and they are independent of x.

$E_1, E_2, N : [0,T] \times P \to X$, with $N(t,\varphi) = \varphi(0) + E_2(t,\varphi)$ are appropriate functions. The control function $u(\cdot) \in L^2([0,T], U)$, the set of all admissible control functions, where U is a reflexive Banach space and $B : U \to X$ is a linear continuous operator.

We consider the following assumptions to prove our main results:

(H1) (i) The operator $\psi : (\cdot,\cdot) : D(A) \to X$ is bounded and continuous, i.e.,

$$\|\psi(t,s)s\| \le h_1\|x\|_{D(A)}, \quad (t,s) \in \Omega \quad h_1 > 0.$$

(ii) There exists $L_\psi > 0$ such that

$$\|\psi(t_2,s)x - \psi(t-1,s)x\| \le L_\psi |t_2 - t_1| \|x\|_{D(A)}$$

for all $x \in D(A), 0 \le s \le t_1 \le t_2 \le T$.

(H2) The closure of $\{R(t,s)[E_1(s,x(s)) + Bu(s)] : (t,s) \in \Omega, \|x(s)\| \le r\}$ is compact.

(H3) The function $E_1 : I \times X \to X$ is continuous for all $t \in I$ and strongly measurable for each $x \in X$. In addition there is $v_r(\cdot) \in L^2(I, R^+)$ for each $r > 0$ such that $\sup\{\|E_1(t,x(t))\| : \|x(t)\| \le r\} \le v_r(t)$ and

$$\lim_{r\to\infty} \inf \frac{1}{r}\|v_r\|_{L^2} = \rho < \infty.$$

(H4) The function $E_2 : I \times P \to X$ is continuous and satisfying

(i) The set $\{E_2(t,\psi) : \|\psi\|_P \le \delta\}$ is equicontinuos on I and is relatively compact in X, where $\delta > 0$.

(ii) There exist $r_1, r_2 > 0$ such that

$$\|E_2(t,\psi)\| \le r_1 + r_2\|\psi\|_P \quad \forall\, t \in I, \quad \text{and} \quad \psi \in P.$$

(iii) There exists $l_2 > 0$ such that

$$E_2(t,\psi_1) - E_2(s,\psi_2) \le l_2\|\psi_1 - \psi_2\| \quad \forall\, t,s \in I, \quad \text{and} \quad \psi_1,\psi_2 \in P.$$

(H5) The control $u(\cdot) = u(t,x) \in L^2(I,U)$ is continuous on I and bounded for every $x \in X$, i.e., $\exists\lambda > 0$ such that $\|u(t)\| = \|u(t,x)\| \leq \lambda\|x\|$.

For each $r > 0$, define the set $\mathrm{B}_r = \{x \in K(T) : \|x(t)\| \leq r \ \forall \ t \in I\}$, where $K(T) = \{x \in C(I,X) | x(0) = \mu(0)\}$ is a convex closed subset of $C(I,X)$.

Theorem 1. *If the assumptions (H1) -(H5) hold, then the system (1) has a mild solution in some B_r, if*

$$2M_1 + M_2 + r_2K_2(t-\nu) + M_1T\rho + M_1\|B\|\lambda T < 1.$$

Let $V\left(\epsilon, \Gamma_0^T\right) = \left(\epsilon I + \Gamma_0^T\right)^{-1}$, $\epsilon > 0$, (The resolvent operator of Γ_0^T)

Consider the associated linear system to (1):

$$\begin{cases} \dfrac{d^2}{dt^2}N(t,x_t) = A(t)N(t,x_t) + \displaystyle\int_0^t \psi(t,s)N(s,x_s)ds + Bu(t), \ t \in I, \\ x_0(t) = \mu(t) \in P, \quad t \in I_0 = (-\infty, 0], \\ x'(0) = x_1 \in X, \end{cases} \tag{8}$$

Lemma 2. *The linear control system (2) is approximately controllable on I if and only if $\epsilon V\left(\epsilon, \Gamma_0^T\right) \to 0$ as $\epsilon \to 0^+$ in the strong operator topology.*

Theorem 2. *Suppose that assumptions (H1) -(H5) hold, and the functions E_1, E_2, are uniformly bounded. If the associated linear system (2) is approximately controllable, then (1) is approximately controllable on I.*

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Inverse problem for Burgers –type parabolic equation

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The work considers the inverse problem of determining an unknown function on the right side of a parabolic equation of Burgers type [1,2]. The unknown function depends on a time variable and the additional condition is specified in integral form. We consider the inverse problem of determining the pair $\{f(t), u(x,t)\}$ from the conditions

$$u_t - u_{xx} + uu_x = f(t)g(x),\ (x,t) \in D = (0,1) \times (0,T], \tag{1}$$

$$u(x,0) = \varphi(x),\ x \in [0,1], \tag{2}$$

$$u(0,t) = \psi_0(t),\ u(1,t) = \psi_1(t),\ t \in [0,T], \tag{3}$$

$$\int\limits_0^1 u(x,t)dx = h(t), t \in [0,T], \tag{4}$$

here $0 < T = const$, the given functions $g(x), \varphi(x), \psi_0(x), \psi_1(x), h(t)$ satisfy the conditions:

A. $g(x) \in C^\alpha[0,1], \int\limits_0^1 g(x)dx = g_0 \neq 0$, $\varphi(x) \in C^{2+\alpha}[0,1]$,

$h(t), \psi_0(t), \psi_1(t) \in C^{1+\alpha}[0,T]$, $\varphi(0) = \psi_0(0)$, $\varphi(1) = \psi_1(0)$, $\alpha \in (0,1)$.

Definition 1. The pair $\{f(t), u(x,t)\}$ is called a classical solution to problem (1)-(4), if :

1)$f(t) \in C^\alpha[0,T]$; 2) $u(x,t) \in C^{2+\alpha,1+\alpha/2}(\overline{D})$; 3)for these functions conditions (1)-(4) are satisfied in the usual way .

It is proved that problem (1)-(4) is equivalent to the problem of determining $\{f(t), u(x,t)\}$ from conditions (1)-(3) and

$$f(t) = \left[h_t(t) - u_x(1,t) + u_x(0,t) + \frac{1}{2}\left(\psi_1^2(t) - \psi_0^2(t)\right)\right]/g_0, \tag{5}$$

Let's call problem (1)-(3),(5) problem B.

Let two sets of data $\{g_i(x), \varphi_i(x), \psi_{0i}(t), \psi_{1i}, h_i(t) i = 1,2\}$ be given. Let us denote task B for these data respectively, by $B1$ and $B2$, and the solutions to these problems by $\{f_1(t), u_1(x,t)\}$, $\{f_2(t), u_2(x,t)\}$.

Theorem 1. *Let: 1) functions $\{g_i(x), \varphi_i(x), \psi_{0i}(t), \psi_{1i}(t), h_i(t) i = 1,2\}$ satisfy condition A; 2) problems $B1$ and $B2$ have solutions in the sense of definition 1 and these solutions belong to the set*

$$K_\alpha = \{(f,u)|f(t) \in C^\alpha[0,T],\ |f(t)| \le c_1, t \in [0,T], u(x,t) \in C^{2+\alpha,1+1/2}(\overline{D}),$$

$$|u|, |u_x|, |u_{xx}| \le c_2,\ (x,t) \in \overline{D},\ c_1, c_2 = \cos t > 0\big\}.$$

Then there exists $T^*(0 < T^* \le T)$, such that for $(x,t) \in [0,1] \times [0,T^*]$ the solutions to problem B is unique and the stability estimate is correct:

$$\|u_1 - u_2\|_D^{(0)} + \|f_1 - f_2\|_{T^*} \le c_3 \Big[\|g_1 - g_2\|_{[0,1]}^{(0)} +$$

$$+ \|\varphi_1 - \varphi_2\|_{[0,1]}^{(2)} + \|\psi_{01} - \psi_{02}\|_{T^*} + \|\psi_{11} - \psi_{12}\|_{T^*} + \|h_1 - h_2\|_{T^*}]$$

here the constant $c_3 > 0$ depends on the data of tasks $B1, B2$ and the set K_α

$$\|z_1 - z_2\|_M^{(l)} = \sup_M \sum_{j=0}^{l} \left|z_{1x}^{(j)} - z_{2x}^{(j)}\right|, \|q_1 - q_2\|_T^{(l)} = \sup_{[0,T]} \sum_{j=0}^{l} \left|q_{1t}^{(j)} - z_{2t}^{(j)}\right|.$$

It can be shown that if there is a solution to problem B in the sense of definition 1, then the function $u(x,t)$ can be represented in the following form

$$u(x,t) = F(x,t) + \int_0^t \int_0^l G(x,t,\xi,\tau)\left[f(\tau)g(\xi) + F_{\xi\xi}(\xi,\tau) - \right.$$

$$\left. - F_\tau(\xi,\tau) - u \cdot u_\xi\right] d\xi d\tau, \tag{6}$$

$$F(x,t) = \varphi(x) + (1-x)[\psi_0(t) - \psi_0(0)] + x[\psi_1(t) - \psi_1(0)],$$

$G(\cdot)$ -fundamental solution of the equation $u_t - u_{xx} = 0$.

Definition 2. We call a pair $\{f(t), u(x,t)\}$ a generalized solution to problem B , is 1) $f(t) \in C[0,T]$; 2) $u(x,t) \in C^{1,0}(\overline{D})$; 3) these function satisfy the system (5)-(6).

Theorem 2. *Let the functions $\{g(x), \varphi(x), \psi_0(t), \psi_1(t), h(t)\}$ satisfy conditions A. Then there exists $T_1 (0 < T_1 \leq T)$, such that for $(x,t) \in [0,1] \times [0,T_1]$ at least one solution to problem B in the sense of definition 2 in the domain $D_1 = [0,1] \times [0,T_1]$.*

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On a fourth-order spectral problem with indefinite weight

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We consider the following eigenvalue problem

$$(p\,(x)y''(x))'' + (q(x)y')(x)' = \lambda r(x)y(x),\ x \in (0,1), \tag{1}$$

$$\begin{array}{l} y'(0)\cos\alpha - (py'')(0)\sin\alpha = 0, \\ y(0)\cos\beta + Ty(0)\sin\beta = 0, \\ y'(1)\cos\gamma + (py'')(1)\sin\gamma = 0, \\ y(1)\cos\delta - Ty(1)\sin\delta = 0, \end{array} \tag{2}$$

where $\lambda \in \mathbb{R}$ is a spectral parameter, $Ty \equiv (py'')' - qy'$, $p \in C^2([0,1];(0,+\infty))$, $q \in C^1([0,1];[0,+\infty))$, $r \in C([0,1];\mathbb{R})$ is an indefinite weight function, i.e., there exists ζ, $\xi \in [0,1]$ such that $r(\zeta)r(\xi) < 0$, α, β, γ, δ are real constants such that $0 \leq \alpha$, β, γ, $\delta \leq \frac{\pi}{2}$ with except the cases $\alpha = \gamma = 0$, $\beta = \delta = \pi/2$ and $\alpha = \beta = \gamma = \delta = \pi/2$ if $q(x) \not\equiv 0$, and three of the parameters α, β, γ, δ are equal to $\pi/2$ if $q \equiv 0$ (see [1]).

Problems of type (1)-(5) arise when describing traveling waves in a suspension bridge, static bending of an elastic plate in a liquid, image processing, and processes of filtration of barotropic gas through a porous medium (see [2]-[4]).

Problem (1)-(5) was previously considered in [5], where it was shown that this problem has two unbounded sequences of real eigenvalues

$$0 < \lambda_1^+ < \lambda_2^+ \leq \ldots \leq \lambda_k^+ \leq \ldots$$

and

$$0 > \lambda_1^- > \lambda_2^- \geq \ldots \geq \lambda_k^- \geq \ldots,$$

and no other eigenvalues. Moreover, λ_1^+ and λ_1^- are simple principal eigenvalues, i.e. the corresponding eigenfunctions $y_1^+(x)$ and $y_1^-(x)$ have no zeros in the interval $(0,1)$.

In this note, we obtained a more general result.

Theorem 1. *All eigenvalues of problem (1)-(5) are simple, and consequently,*

$$0 < \lambda_1^+ < \lambda_2^+ < \ldots < \lambda_k^+ < \ldots$$

and

$$0 > \lambda_1^- > \lambda_2^- > \ldots > \lambda_k^- > \ldots .$$

Moreover, for each $k \in \mathbb{N}$, the eigenfunction $y_k^+(x)$ and $y_k^-(x)$ corresponding to the eigenvalues λ_k^+ and λ_k^-, respectively, have exactly $k-1$ simple zeros in $(0,1)$.

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Some global results for nonlinearizable one dimensional Dirac systems

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Consider following the nonlinear Dirac system

$$(\ell w)(x) \equiv Bw'(x) - P(x)w(x) = \lambda w(x) + h(x, w(x), \lambda),\ 0 < x < \pi, \tag{1}$$

$$\begin{aligned} U_1(\lambda, w) := (\lambda \cos\alpha + a_0, \lambda \sin\alpha + b_0)w(0) = \\ (\lambda \cos\alpha + a_0)\, v(0) + (\lambda \sin\alpha + b_0)\, u\,(0) = 0, \end{aligned} \tag{2}$$

$$\begin{aligned} U_2(\lambda, w) := (\lambda \cos\beta + a_1, \lambda \sin\beta + b_1)w(\pi) = \\ (\lambda \cos\beta + a_\pi)\, v(0) + (\lambda \sin\beta + b_0)\, u\,(\pi) = 0, \end{aligned} \tag{3}$$

where

$$B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ P(x) = \begin{pmatrix} p\,(x) & 0 \\ 0 & r(x) \end{pmatrix},\ w(x) = \begin{pmatrix} u(x) \\ v(x) \end{pmatrix},$$

$\lambda \in \mathbb{C}$ is an eigenvalue parameter, $p\,(x)$ and $r(x)$ are real-valued continuous functions on $[0, \pi]$, α, β, a_0, b_0, a_1 and b_1 are real constants such that

$$0 \le \alpha,\, \beta < \pi$$

and
$$\sigma_0 = a_0 \sin\alpha - b_0 \cos\alpha < 0, \;\; \sigma_1 = a_1 \sin\beta - b_1 \cos\beta > 0. \tag{4}$$

Moreover, the nonlinear term h has the form $f+g$, where $f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$ and $g = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$ are real-valued continuous functions on $[0,\pi] \times \mathbb{R}^2 \times \mathbb{R}$ and satisfy the following conditions:

there are positive constants K and L such that

$$|f_1(x,w,\lambda)| \le K|w|, \;\; |f_2(x,w,\lambda)| \le L|w|, \;\; (x,w,\lambda) \in [0,\pi] \times \mathbb{R}^2 \times \mathbb{R}; \tag{5}$$

for every bounded interval $\Lambda \subset \mathbb{R}$,

$$g(x,w,\lambda) = o(|w|) \text{ as } |w| \to 0, \tag{6}$$

and
$$g(x,w,\lambda) = o(|w|) \text{ as } |w| \to \infty, \tag{7}$$

uniformly for $(x,\lambda) \in [0,\pi] \times \Lambda$ (here we denote by $|\cdot|$ the Euclidean norm in $\mathbb{R}^2$).

The global bifurcation of nontrivial solutions to problem (1)-(3) when conditions (5), (6) are satisfied is studied in [3], and when conditions (5), (7) are satisfied, it is studied in [2, 4]. In these papers proved the existence of global components of nontrivial solutions to problem (1)-(3) branching from zero and infinity and having the nodal properties of eigenvector-functions of the linear problem
$$\begin{cases} (\ell w)(x) = \lambda w(x), \, x \in (0,\pi), \\ U_1(\lambda, w) = 0, \\ U_2(\lambda, w) = 0, \end{cases} \tag{8}$$

that obtained from (1)-(3) by setting $h \equiv 0$ in some neighborhoods of bifurcation points and intervals.

Note that problem (8) was studied in [1], where it was shown that the eigenvalues λ_k, $k \in \mathbb{Z}$, of this problem are real and simple, and they can be numbered in ascending order on the real axis

$$\ldots < \lambda_{-k} < \ldots < \lambda_{-2} < \lambda_{-1} < \lambda_0 < \lambda_1 < \lambda_2 < \ldots < \lambda_k < \ldots .$$

Let $E = C([0,\pi];\mathbb{R}^2)$ be the real Banach space with the usual norm

$$||w|| = \max_{x\in[0,\pi]} |u(x)| + \max_{x\in[0,\pi]} |\vartheta(x)|.$$

For each $k \in \mathbb{N}$, $k \le m_{-1}$ and $k \ge m_1$ (regarding the definition of numbers m_{-1} and m_1 see [4, § 2]) and each $\nu \in \{+,-\}$ let S_k^ν denote the class of vector functions $w \in E$, constructed in [4, § 2], which has the oscillatory properties of eigenvector-function w_k of the linear problem (8).

Let $\mathcal{D}$ be the closure of the set of nontrivial solutions of problem (1)-(3).

Theorem 1. *Let conditions (4)-(7) hold. Then there exit unbounded in $\mathbb{R} \times E$ connected components $\mathcal{Q}_k^\nu$ and $\mathfrak{Q}_k^\nu$ of the set $\mathcal{D}$ such that*

(i) $(I_k \times \{0\}) \subset \mathcal{Q}_k^\nu$ and $(I_k \times \{\infty\}) \subset \mathfrak{Q}_k^\nu$;

(ii) $\mathcal{Q}_k^\nu \subset (\mathbb{R} \times S_k^\nu) \cup (I_k \times \{0\})$, $\mathfrak{Q}_k^\nu \subset (\mathbb{R} \times S_k^\nu) \cup (I_k \times \{\infty\})$;

(iii) either $\mathfrak{Q}_k^\nu$ meets $(\lambda, 0)$ for some $\lambda \in \mathbb{R}$, or the projection of this set onto $\mathbb{R} \times \{0\}$ is unbounded.

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Existence of strong solutions to second order semilinear elliptic systems

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In the paper, we establish sufficient conditions for the existence of a strong solution to a boundary value problem for a second order semilinear elliptic system of equations with the summable right side (see [1]).

Let Ω be a bounded domain located in N dimensional Euclidean space R^N of the points $x = (x_1, ..., x_N)$ with the boundary $\partial\Omega$ of the class $c^2, u = u(x), Du = Du(x)$ be a gradient of the functions $u(x), \overrightarrow{u}(x) = (u_1(x), ..., u_n(x))$ be a vector from the functions $u_j(x)$ and $D\overrightarrow{u} = (Du_1, ..., Du_n)$. We will write that $\overrightarrow{u} \in \overrightarrow{L}_p(\Omega)$ if $u_j \in L_p(\Omega)$ for each $j = 1, .., n$ and $\|\overrightarrow{u}\|_{p;\Omega} = \sum\limits_{j=1}^{n} \|u_j\|_{Lp(\Omega)}$. We will adhere to similar arguments in the case $\overrightarrow{L}_q(\Omega)$, $\overrightarrow{W}_p^2(\Omega)$ and $\overrightarrow{C}(\Omega)$ as well.

We are given the linear operator

$$\overrightarrow{L}\overrightarrow{u} = \left\{ \sum_{|\alpha|\le 2} \sum_{j=1}^{n} a_{\alpha j}^k(x) D^\alpha u_j, \quad k = \overline{1,n} \right\},$$

where $a_{\alpha j}^k(x)$ are the known real and rather smooth functions on Ω.

Let us consider the boundary value problem

$$\left\{ \begin{array}{ll} \overrightarrow{L}\overrightarrow{u} = \overline{f(x, \overline{u})}, & x \in \Omega, \\ \overline{u}|_{\partial\Omega} = \overline{\varphi}(x), & x \in \partial\Omega, \\ \overrightarrow{\varphi}(x) \in \overline{W}_p^{2-1/p}(\partial\Omega). & \end{array} \right. \tag{1}$$

This problem is studied in the class of real functions from the Sobolev space $\overrightarrow{w}_p^2(\Omega)$ for $p > 1$ and, $N \ge 3$.

The following conditions are assumed to be fulfilled:

A 1) The operator $\overrightarrow{L}$ is a uniformly elliptic operator of second order.

A 2) There is a priori estimate $\|\overrightarrow{u}\|_{l;\Omega} \leq M$ with some $1 \leq l < \infty$ and, $M > 0$ is a constant

A 3) The functions $f_k(x, \overline{\xi}), k = \overline{1, n}$ that compile the vector f is determined on $\overrightarrow{\Omega} \times R^n$and is Caratheodorian

A 4) the following inequality is fulfilled for all $(x, \overrightarrow{\xi}) \in \overrightarrow{\Omega} \times R^n$

$$\overrightarrow{n}\sum_{k=1} \left|f_k(x, \overline{\xi})\right| \leq b(x) + b_0(x) \left|\overrightarrow{\xi}\right|^{\mu},$$

where $b(x) \in L_p(\Omega),\ b_0(x) \in L_q(\Omega),\ b(x) \geq 0,\ b_0(x) \geq 0,\ \max\left\{\frac{N}{2}, p\right\} < q < \infty,\ N \geq 3,\ 1 < p < \infty$

B 5) $p^* < p < \infty$, where

$$p^* = \begin{cases} 1, & 1 \leq l \leq \frac{N}{N-2}, \\ \frac{lN}{N+2l}, & l > \frac{N}{n-2}. \end{cases}$$

B 6)

$$\mu \leq 1 + l\left(\frac{2}{N} - \frac{1}{q}\right) \equiv \mu^*.$$

B 7) If $\mu = \mu^*$, then

$$\forall \varepsilon > 0, \|b_0\|_{q:\Omega} \|\overrightarrow{u}\|_{l;\Omega}^{\mu-1} < \varepsilon.$$

When studying the problem (1), the theorem on embedding of anisotropic spaces $\overrightarrow{W}_p^2(\Omega)$ and the multiplicative inequality that follow from it are essentially used.

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Inverse nodal problems for singular diffusion equation

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Solvable models of quantum mechanics are investigated in detail in the study [1]. As can be seen, these models are generally expressed with Hamilton operators or Schrödinger operators with singular coefficients. Many of the problems expressed by these models are related to the solution of spectral inverse problems for differential operators with singular coefficients. However, many problems in mathematical physics are reduced to the study of differential operators whose coefficients are generalized functions.

We consider the following quadratic pencils of Sturm-Liouville equation of the form

$$\ell y := -y^{''} + \left[\lambda p\left(x\right) + q(x)\right] y = \lambda^2 y, \qquad x \in [0, \pi] \setminus \{a\}\,, \tag{1}$$

with the boundary conditions

$$U\left(y\right) := y^{'}\left(0\right) - hy(0) = 0, \tag{2}$$

$$V\left(y\right) := y^{'}\left(\pi\right) + Hy(\pi) = 0, \tag{3}$$

where $q(x)$ is a real function belonging to the space $L_2\left[0, \pi\right], \lambda$ is a spectral parameter, $p\left(x\right) = \beta\delta\left(x - a\right)$, h, a, H and β are real numbers.

In this paper, unlike previous studies on this subject, when the discontinuity point $a \in (0, \pi)$ is any of the contable number of irrational points in the form of $a_r = r\pi$, $(r \in (0,1) \cap \mathbb{Q})$, the proof of the uniqueness theorem is given for the solution of the inverse nodal problem and give an algorithm for the reconstruction of the coefficients of the problem using asymptotics of the nodal points.

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Inverse problems for the subdiffusion equation with fractional Caputo derivative

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This is a joint work with M. Shakarova, Institute of Mathematics, Uzbekistan (see [1], [2], [3]).

In the Hilbert space H consider the subdiffusion equation $D_t^\rho u(t)+Au(t)=fg(t)$, $\rho\in(0,1]$, $t\in(0,T]$. In this equation $g(t)\in C[0,T]$, $f\in H$, A is an unbounded positive self-adjoint operator and D_t^ρ stands for the Caputo fractional derivative. We assume that A has a compact inverse A^{-1}.

Let us consider two types of initial conditions (a) $u(0)=\varphi$, (b) $u(0)=u(T)$, where $\varphi\in H$ and obtain two (let us call them (a), (b), respectively) forward problems.

First, we prove the existence and uniqueness of a solution to both forward problems. Next we consider three inverse problems (IP1 -IP3) of determining a pair $\{u(t),f\}$. IP1 := Problem (a) + additional condition: $u(t_0)=\Psi$, $t_0\in(0,T]$, IP2 := Problem (b) + additional condition: $u(t_0)=\Psi$, $t_0\in(0,T)$, and IP3 := Problem (a) + additional condition: $\int\limits_0^T u(t)dt=\psi$, where ψ and Ψ are given elements of H.

We briefly present the results obtained for the above inverse problems. A criterion for the uniqueness of a solution to the inverse problems is found. It is interesting to note that this criterion is completely different for each problem. But if the function $g(t)$ retains its sign, then all criteria are met and we were able to prove the existence and uniqueness of a solution to inverse problems in this case. For all inverse problems, examples of functions $g(t)$ have been constructed (different for each problem), changing sign and, as a result, there is no unique solution to the inverse problem. When $g(t)$ changes sign, then in some cases, the existence and uniqueness of the solution were proved (for all 3 problems), while in other cases, the necessary and sufficient condition for the existence of the solution was found (different for each problem). We note that almost all the results obtained here are also new for the classical diffusion equation.

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On the boundedness of the resolvent of the operator generated by partial operator-differential expressions of higher order in Hilbert space

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Let $H_0, H_1, ...H_{2m}$ be Hilbert spaces, $H_{i+1} \subset H_i, i = 0, 1, 2, ..., 2m-1$, where all embeddings are compact.

We consider the following differential expression

$$L(x, D)u = \sum_{|\alpha| \leq 2m} A_\alpha(x) D^2 u, \quad x \in R^n$$

where

$$\alpha = (\alpha_1, \alpha_2, ..., \alpha_n), |\alpha| = \alpha_1 + \alpha_2 + ... + \alpha_n, \;\; D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} ... \partial x_n^{\alpha_n}}.$$

The function $u(x) \in H_{2m}$ is such that $D^\alpha u \in H_{2m-|\alpha|}$. It is assumed that for each $x \in R^n A_\alpha(x) : H_{2m-|\alpha|} \to H_0$ is a bounded operator $A_0(x) = A_0 +$

$\gamma(x)E$, $A_0 : H_0 \to H_0$ is such a positive definite self-adjint operator that A_0^{-1} is completely continuous. The complex-valued function is assumed to be measurable and locally bounded.

It is assumed that the coefficients of the operator $L(x, D)$ satisfy the following conditions.

Denote $R_0(x,\xi) = \left[\sum\limits_{|\alpha|=2m} A_\alpha(x)(i\xi)^\alpha\right]^{-1,}$, $\xi = (\xi_1, \xi_2, ..., \xi_n)$, $\xi^\alpha = \xi_1^{\alpha_1}, \xi_2^{\alpha_2}, ..., \xi_{n1}^{\alpha_n}$.

I. $R_0(x,\xi)$for all $\xi \in R^n/0$ is a bounded operator $H_0 \to H_{2m}$, moreover

$$\sum_{i=0}^{2m} |\xi|^i \|R_0\|_{H_0\to H_{2m-i}} \leq \delta_1,$$

where $\delta_1 = const$ is independent of x, ξ.

II . There exists such a ray $l = \{\lambda : \arg\lambda = \beta\}$ of such a complex plane λ that the operator

$R(x,\xi,\lambda) = \left[\sum\limits_{|\alpha|\leq 2m} A_\alpha(x)(i\xi)^\alpha - \lambda E\right]^{-1,}$ is a bounded operator $H_0 \to H_{2m}$ for $\lambda \in l$, $\xi \in R^n$, $|x| > c$ and

$$(|\gamma(x)| + |\lambda|)\|R_0\|_{H_0\to H_{2m}} + \sum_{i<2m} |\xi|^{2m-i} \|R_0\|_{H_0\to H_{2m-i}} \leq \delta_2.$$

III. The quantities $\sup\limits_{\|x-x_0\|\leq h} \|A_\alpha(x) - A_\alpha(x_0)\|$, $\sup\left|\frac{\gamma(x)-\gamma(x_0)}{\gamma(x_0)}\right|$ tend to zero as $h \to 0$ uniformly with respect to n, x_0, $|\alpha| = 2m$.

IV. $\sum\limits_{0<|\alpha|\leq 2m} \|A_\alpha(x)\| < \delta_3$.

Denote by H_ia space with a scalar product

$$(f,g)_{H_i} = \int_{R^n} (f(x), g(x))_{H_i} dx, f, g \in H_i, i = 0, 1, ..., 2m.$$

The following theorems are the main results of this paper.

Theorem 1. *Let conditions I-IV be fulfilled, and*

$$\sum_{|\alpha|\leq 2m} \int_{R^n} \|D^\alpha u\|_{H_0}^2 dx + \int_{R^n} \gamma^2(x) \|u\|_{H_0}^2 dx < \infty$$

for $\lambda \in l$, $|\lambda| \geq \lambda_0$, *then*

$$\sum_{|\alpha| \leq 2m} \|D^\alpha u\|_{H_0}^2 \, dx \leq c_1 \, \|(L - \lambda E) \, u\|_{H_0}^2$$

where $c_1 = const$ *is independents of* λ.

Theorem 2. *If the conditions of theorem 1 are fulfilled, then the operator* $(L - \lambda E)^{-1} : H_0 \to H_0$ *is a bounded operator for each* $\lambda \in l$, $|\lambda| \geq \lambda_0$.

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Absence of global solutions to a system of high-order semilinear equations with a biharmonic operator in the main part

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Let's denote: $x = (x_1, ..., x_n) \in R^n$, $r = |x| = \sqrt{x_1^2 + ... + x_n^2}$, $R = const > 0$, $B_R = \{x : |x| < R\}$, $B'_R = \{x : |x| > R\}$, $\partial B_R = \{x : |x| = R\}$, $B_{R_1,R_2} = \{x : R_1 < |x| < R_2\}$, $\bar{B}'_R = R^n \backslash B_R$, $Q_R = B_R \times (0; +\infty)$, $Q'_R = B'_R \times (0; +\infty)$, $C^{4,k}_{x,t}(Q'_R)$ is the set of functions that are four times continuously differentiable with respect to x and k times continuously differentiable with respect to t in Q'_R.

In the domain Q'_R we consider the system of equations

$$\begin{array}{l} \frac{\partial^k u_1}{\partial t^k} + \Delta^2 u_1 - \frac{C_1}{|x|^4} u_1 = |x|^{\sigma_1} |u_2|^{q_1} \\ \frac{\partial^k u_2}{\partial t^k} + \Delta^2 u_2 - \frac{C_2}{|x|^4} u_2 = |x|^{\sigma_2} |u_1|^{q_2} \end{array} \tag{1}$$

with the initial condition

$$u_i^{(m)}\Big|_{t=0} = u_{i0}^m(x) \geq 0, \ u_{i0}^{k-1}(x) \geq 0 \tag{2}$$

and the conditions

$$\int_0^T \int_{\partial B_R} u_i dx dt \geq 0, \ \ \int_0^T \int_{\partial B_R} \Delta u_i dx dt \leq 0, \ \forall T > 0, \tag{3}$$

where $n > 4, \ q_i > 1, \ k \geq 1 \ \sigma_i \in R, \ 0 \leq C_i < \left(\frac{n(n-4)}{4}\right)^2, \ u_{i0}^m(x) \in C(B'_R)$,

$$\Delta^2 u = \Delta(\Delta u), \ i = 1, 2, \ m = 0, 1, ..., k-1.$$

Note that the case $C_i = 0$ was considered in work (1), and the case $C_i \neq 0, i = 1, 2$, but $k = 1$ in work (2).

We will study the nonexistence of a global solution of problem (1)-(3). By a global solution of problem (1)-(3) we understand a pair of functions (u_1, u_2), such that $u_1(x,t), u_2(x,t) \in C^{4,k}_{x,t}(Q'_R) \bigcap C^{3,k-1}_{x,t}(\bar{B}'_R \times [0,+\infty))$ and satisfy the system (1) at every point of Q'_R, the initial condition (2) and conditions (3).

The avoid complications, we introduce the following denotation:

$$D_i = \sqrt{(n-2)^2 + C_i}, \quad \lambda_i^{\pm} = \sqrt{\left(\frac{n-2}{2}\right)^2 + 1 \pm D_i},$$

$$\alpha_1 = \frac{\lambda_1^- + \sigma_1 + \frac{n-4}{2} + \frac{4}{k}}{\lambda_2^- + \frac{n-4}{2} + \frac{4}{k}}, \alpha_2 = \frac{\lambda_2^- + \sigma_2 + \frac{n-4}{2} + \frac{4}{k}}{\lambda_1^- + \frac{n-4}{2} + \frac{4}{k}},$$

$$\beta_1 = \frac{\lambda_1^- + \sigma_1 + \frac{n+4}{2} + \frac{4}{k}}{\lambda_2^- + \frac{n-4}{2}}, \beta_2 = \frac{\lambda_2^- + \sigma_2 + \frac{n+4}{2} + \frac{4}{k}}{\lambda_1^- + \frac{n-4}{2}},$$

$$\theta_1 = \frac{\sigma_1 + 4 + q_1(\sigma_2 + 4)}{q_1 q_2 - 1} - \lambda_1^- - \frac{n-4}{2} - \frac{4}{k},$$

$$\theta_2 = \frac{\sigma_2 + 4 + q_2(\sigma_1 + 4)}{q_1 q_2 - 1} - \lambda_2^- - \frac{n-4}{2} - \frac{4}{k}, \quad i = 1, 2.$$

The main result of this paper reads as follows.

Theorem. *Assume that* $n > 4$, $\beta_i > 1$, $0 \le C_i < \left(\frac{n(n-4)}{4}\right)^2$ *and* $1 < q_i \le \beta_i$, $\max(\theta_1, \theta_2) \ge 0$, $(q_1, q_2) \ne (\alpha_1, \beta_2)$ *in case* $\alpha_1 > 1$, $(q_1, q_2) \ne (\beta_1, \alpha_2)$ *in case* $\alpha_2 > 1$, $i = 1, 2$. *Then there is no nontrivial global solution of (1)-(3).*

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On solvability problems of elliptic equations in Banach-Sobolev spaces

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In this work, we consider the Banach-Sobolev space $W_X^m(\Omega)$ of differentiable functions (in the Sobolev sense), generated by norm of the Banach Function Space (BFS) $X(\Omega)$ on the measure space $(\Omega; dx)$ with Lebesgue measure. Three approaches to defining the concepts of trace, trace operator, and trace space with respect to the space $W_X^m(\Omega)$ are given. These concepts allow us to consider the boundary value problems (in particular, the Dirichlet problem) for uniformly elliptic equations of m-th order in the spaces $W_X^m(\Omega)$ (corresponding solution is a strong solution). The correct solvability of the Dirichlet problem for a polyharmonic equation in the Sobolev space $W_X^{2m}(\Omega)$, where X is a symmetric space, is separately established. This class of spaces includes the Lebesgue space $L_p(\Omega)$, the grand Lebesgue space $L_{p)}(\Omega)$, the Orlicz space $L_M(\Omega)$, the Marchinkiewichz space $M_{p;\lambda}(\Omega)$ and others. The obtained results are new even for these concrete spaces.

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Valuation of corporate securities with finite maturity debt: A probabilistic approach

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We provide explicit valuation formulas for debt and equity in a model with endogenous default time and finite debt maturity date. The equity value has an early default premium representation where the endogenous default boundary solves a nonlinear Volterra integral equation. The debt value has a representation with a component tied to the local time of the cash flow process along the default boundary and parameterized by the delta of the debt along the boundary. The latter satisfies linear Volterra equations of the first and second kind. Algorithms for numerical implementation are provided.

1 Model

Let us assume that the firm at time t produces the before-tax cash flow at rate X_t that follows

$$dX_t/X_t = \mu dt + \sigma dW_t \tag{9}$$

under the risk-neutral measure Q, where $\mu = r - \delta < r$ is the risk-neutral drift, $\delta > 0$ is the payout ratio, and $\sigma > 0$ is the volatility parameter. The process W is the standard Brownian motion under Q defined on the probability space $(\Omega, \mathcal{F}, Q)$. For simplicity we assume that all parameters are constant, however all the results can be extended to the case of time-dependent parameters.

We assume that the firm issues two types of securities at $t = 0$: equity and debt. The latter is given in the form of the coupon-bearing bond that pays a coupon c continuously through time until the maturity date $T > 0$, at which there is a principal repayment $P > 0$, and after which the firm becomes an all-equity firm with after-tax value

$$U(X_T) = (1-\theta)\mathbb{E}_T\left[\int_T^\infty e^{-r(s-T)} X_s ds\right] = (1-\theta)X_T/\delta. \tag{10}$$

The parameter $\theta \in (0,1)$ is the tax rate, and we suppose the debt is not callable and cannot be prepaid before the maturity time T. Equityholders solve the following optimal default problem

$$E(X_0; c, P) = \sup_{\tau\in[0,T]} \mathbb{E}\left[\int_0^\tau e^{-rt}(1-\theta)(X_t - c)dt + e^{-rT}\left(U(X(T)) - P\right)^+ I(\tau = T)\right] \tag{11}$$

where the supremum is taken over the set of stopping times with values in $[0, T]$ and $I_{\tau=T}$ is the indicator of the event the default has not happened before T. The intuition behind (11) is the following. Until the default time τ, the equityholders collect the cash flow X_t and pay the coupon c. If they do not default before T, they have the option to extend their ownership and get the all-equity value of the firm $U(X_T)$, but for this, must pay the face value of the debt P. If $\tau < T$, the firm's assets are transferred from equityholders to debtholders. This formulation is consistent with the model in [2], which was formulated for the case of a consol bond.

Given the optimal default (random) time τ_d in (11), we define the debt value as

$$D(X_0; c, P) = \mathbb{E}\Bigg[\int_0^{\tau_d} e^{-rt} c dt + e^{-r\tau_d}(1-\alpha)U(X_\tau)I(\tau_d < T) \qquad (12)$$

$$+ e^{-rT}\left(P \cdot I(\Omega/A) + (1-\alpha)U(X_T)I(A)\right) \cdot I(\tau_d = T)\Bigg]$$

where $\alpha \in [0,1]$ is the bankruptcy cost and $A := \{U(X_T) < P\}$ is the event of default at time T.

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Reduction of the systems of equations describing the dynamics of oscillations of a suspension bridge to the operator equation

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We consider the following mixed problem for the oscillations of the bridge with strong delay:

$$u_{tt}(x,t)+u_{xxxx}(x,t)+[u-v]_{+}+\lambda_1 u_t(x,t)+\lambda_2 z_1(x,1,t)+$$

$$+g_1(u(x,t),v(x,t))=h_1(x,t), \qquad x\in(0,l),\quad t>0 \tag{1}$$

$$v_{tt}(x,t)-v_{xx}(x,t)-[u-v]_{+}+\mu_1 v_t(x,t)+\mu_2 z_2(x,1,t)+$$

$$+g_2(u(x,t),v(x,t))=h_2(x,t), \qquad x\in(0,l),\quad t>0 \tag{2}$$

$$\tau_1 z_{1t}(x,\rho,t)+z_{1\rho}(x,\rho,t)=0, \qquad \rho\in(0,1),\ x\in(0,l),\ t>0 \tag{3}$$

$$\tau_2 z_{2t}(x,\rho,t)+z_{2\rho}(x,\rho,t)=0, \qquad \rho\in(0,1),\ x\in(0,l),\ t>0 \tag{4}$$

with boundary conditions

$$u(0,t)=u_{xx}(0,t)=u(l,t)=u_{xx}(l,t)=0, \quad t>0 \tag{5}$$

$$v(0,t)=v(l,t)=0, \quad t>0 \tag{6}$$

$$z_i(0,\rho,t)=z_i(l,\rho,t)=0, \quad t>0 \tag{7}$$

$$z_2(0,\rho,t)=z_2(l,\rho,t)=0, \quad t>0 \tag{8}$$

and initial conditions

$$u(x,0)=u_0(x), \quad u_t(x,0)=u_1(x), \quad x\in(0,l), \tag{9}$$

$$z_1(x,\rho,0)=f_1(x,-\rho\tau_1), \quad x\in(0,l),\ \rho\in(0,1), \tag{10}$$

$$v(x,0)=v_0(x), \quad v_t(x,0)=v_1(x), \quad x\in(0,l), \tag{11}$$

$$z_2(x,\rho,0)=f_2(x,-\rho\tau_2), \quad x\in(0,l), \quad \rho\in(0,1), \tag{12}$$

where z_1 and z_2 are defined as follows:

$$z_1\left(x,\rho,t\right)=u_t(x,t-\tau_1\rho),\rho\in(0,1),x\in(0,l),t>0$$

$$z_2\left(x,\rho,t\right)=\upsilon_t(x,t-\tau_2\rho),\rho\in(0,1),x\in(0,l),t>0.$$

z_1 and z_2 are solutions of the problems :

$$\tau_1 z_{1t}\left(x,\rho,t\right)+z_{1\rho}\left(x,\rho,t\right)=0 \quad z_1\left(x,\rho,0\right)=f_1(x,-\rho\tau_1), \tag{13}$$

$$\tau_2 z_{2t}\left(x,\rho,t\right)+z_{2\rho}\left(x,\rho,t\right)=0, \quad z_2\left(x,\rho,0\right)=f_2(x,-\rho\tau_2) \tag{14}$$

where $\rho\in(0,1),\ x\in(0,l),\ t>0.$
The problem (1)-(14) will be studied in the space

$$\mathcal{H}=\widehat{\mathrm{H}}^2\times L_2\left(0,l\right)\times\widehat{\mathrm{H}}^1\times L_2\left(0,l\right)\times L_2((0,1)\times(0,l)) \quad \times L_2((0,1)\times(0,l)).$$

$\mathcal{H}$ is a Hilbert space equipped with the inner product

$$\langle\omega,\widetilde{\omega}\rangle=\int_0^l u_{1xx}\tilde{u}_{1xx}dx+\int_0^l u_2\tilde{u}_2dx+\int_0^l u_3\tilde{u}_3dx+\int_0^l u_4\tilde{u}_4dx+$$

$$+\tau\left|\lambda_2\right|\int_0^l\int_0^1 z_1\tilde{z}_1d\rho dx+\tau\left|\mu_2\right|\int_0^l\int_0^1 z_2\tilde{z}_2d\rho dx,$$

where $\omega=(u_1,\ u_2,u_3,u_4,\ z_1,z_2)^T$, $\widetilde{\omega}=(\ \tilde{u}_1,\ \tilde{u}_2,\ \ \tilde{u}_3,\ \ \tilde{u}_4,\ \tilde{z}_1,\widetilde{z}_2)^T\in\mathcal{H}$.

In the space $\mathcal{H}$, we define a linear operator A_0 :

$$D(A_0)=\Big\{\omega:\omega=(u_1,u_2,u_3,u_4,z_1,z_2)^T\in\mathcal{H},u_1\in\widehat{H}^4,u_2\in\widehat{H}^2,u_3\in\widehat{H}^2,$$

$$u_4\in\widehat{H}^1,z_i,z_{i\rho}\in L_2((0,1)\times L_2\left(0,l\right)),\ z_i\left(\cdot,1\right)\in L_2((0,1)\times(0,l)),\ i=1,2.\Big\},$$

$$A_0\omega=(-u_2,u_{1xxxx}+\lambda_1u_2+\lambda_2z_1\left(\cdot,1\right),$$

$$-u_4,-\ u_{3xx}+\mu_1u_4+\mu_2z_2\left(\cdot,1\right),\ \frac{1}{\tau_1}z_{1\rho},\ \frac{1}{\tau_2}z_{2\rho}),$$

$$\omega=(u_1,u_2,u_3,u_4,z_1,z_2)^T\in D(A_0).$$

We also define the nonlinear operators $A_1(\cdot)$ and $F(\cdot)$, acting from $\mathcal{H}$ to $\mathcal{H}$, respectively:

$$A_1(\omega) = (0, [u_1 - u_3]_+,\ 0,\ -\ [u_1 - u_3]_+, 0\ , 0),$$

$$F(t, \omega) = (0,\ g_1(\cdot, u, v) + h_1(\cdot, t),\ 0,\ g_2(\cdot, u, v) + h_2(\cdot, t),\ 0,\ 0).$$

The problem (1)-(12) can be written as the Cauchy problem in the Hilbert space $\mathcal{H}$:

$$\begin{cases} \omega' + A_0\omega + A_1(\omega) + F(\omega) = 0, \\ \omega(0) = \omega_0, \end{cases} \tag{15}$$

where

$$\omega = \omega(t) = (u_1(t),\ u_2(t), u_3(t), u_4(t),\ z_1(t), z_2(t))^T,$$
$$\omega(0) = \omega_0 = (u_{10},\ u_{20},\ u_{30}, u_{40},\ z_1(\cdot - \rho\tau), z_2(\cdot - \rho\tau)),$$
$$u_1\left(t\right) = u\left(\cdot t\right), u_2\left(t\right) = u_t\left(\cdot t\right), u_3\left(t\right) = v\left(\cdot t\right), u_4\left(t\right) = v_t\left(\cdot t\right),$$
$$z_1(t) = z_1(\cdot, t), z_2(t) = z_2(\cdot, t).$$

Using the methods of operator-differential equations, we obtain the following statement about the existence and uniqueness of the solution to the problem (15):

Theorem 1. *Assume that conditions*

$$\lambda_i,\ \mu_i \in R,\ i = 1, 2$$

and

$$h_i(\cdot) \in W_2^1([0, +\infty)\ ; L_2(0, l)),\ i = 1, 2.$$

are satisfied.
Then for any $\omega_0 \in \mathcal{H}$, there exists $T' > 0$ such that the problem (15) has a unique solution

$$\omega(\cdot) \in \mathrm{C}(\ [0, T']\ ,\ \mathcal{H}\ ).$$

Moreover, if $\omega_0 \in D\left(A_0\right)$, then

$$\omega(\cdot) \in C^1(\ [0, \mathrm{T}']\ ,\ \mathcal{H}\ ) \cap \mathrm{C}(\ [0, T']\ ,\ D\left(A_0\right)).$$

If $T_{max} = \sup \mathrm{T}'$, i.e., T_{max} is the length of the maximal existence interval of the solution $\omega(\cdot) \in \mathrm{C}(\ [0, T_{max})]\ ,\ \mathcal{H}\ )$, then either
(i) $T_{max} = +\infty$, or (ii) $T_{max} \limsup_{t\to+\infty} \|\omega(\mathrm{t})\|_{\mathcal{H}} = +\infty$.

Basis properties of eigenfunctions of a spectral problem all boundary conditions which contain the eigenvalue parameter

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We consider the following spectral problem

$$y^{(4)}(x) = \lambda y(x),\ 0 < x < 1, \tag{1}$$

$$y''(0) = a\lambda y'(0), \tag{2}$$

$$y'''(0) - b\lambda y(0) = 0, \tag{3}$$

$$y''(1) - c\lambda y'(1) = 0, \tag{4}$$

$$y'''(1) - d\lambda y(1) = 0, \tag{5}$$

where $\lambda \in \mathbb{C}$ is a spectral parameter, a, b, c and d are real constants such that

$$a < 0,\ b > 0,\ c > 0 \text{ and } d < 0.$$

Problem (1)-(5) arises when describing small bending vibrations of a homogeneous elastic cantilever beam at both ends of which follower forces act and to these ends are attached loads with weightless rods held in equilibrium by elastic springs. (see [1, p. 152-154]).

Lemma 1. *The eigenvalues of problem (1)-(5) are nonnegative and form a sequence tending to* $+\infty$.

Lemma 2. *The nonzero eigenvalues of problem (1)-(5) are simple.*

Remark 1. Note that $\lambda = 0$ is an eigenvalue of problem (1)-(5) which has a geometric multiplicity of 2 (in this case $\lambda_1 = \lambda_2 = 0$) and the corresponding root subspace consists of functions of the form $y(x) = px + q,\ p,\ q \in \mathbb{R}$, $x \in [0, 1]$. Hence the functions $y_1(x)$ and $y_2(x)$ can be chosen arbitrarily of the form $y_1(x) = p_1x + q_1$ and $y_2(x) = p_2x + q_2$, p_1, p_2, q_1, $q_2 \in \mathbb{R}$.

Theorem 1. *There exists an infinitely nondecreasing sequence* $\{\lambda_k\}_{k=1}^{\infty}$ *of eigenvalues of the boundary value problem (1)-(5) such that* $\lambda_1 = \lambda_2 = 0$ *and* $\lambda_k > 0$, $k \geq 3$.

Remark 2. It follows from Theorem 1 and [2, Lemma 2.1] that for the eigenfunction $y(x,\lambda)$ corresponding to the eigenvalue $\lambda > 0$ of the problem (1)-(5) the relation $y'(0,\lambda) \neq 0$ holds.

Let i, j, r and l be different arbitrary fixed natural numbers and

$$\Delta_{i,j,r,l} = \begin{vmatrix} y_i'(0) & y_i(0) & y_i'(1) & y_i(1) \\ y_j'(0) & y_j(0) & y_j'(1) & y_j(1) \\ y_r'(0) & y_r(0) & y_r'(1) & y_r(1) \\ y_l'(0) & y_l(0) & y_l'(1) & y_l(1) \end{vmatrix}.$$

Theorem 2. *Let i, j, r and l be different arbitrary fixed positive integers. If $\Delta_{i,j,r,l} \neq 0$, then the system $\{y_k(x)\}_{k=1,\, k \neq i,j,r,l}^{\infty}$ is a basis in $L_p(0,1), 1 < p < \infty$, and for $p = 2$ this basis is unconditional. If $\Delta_{i,j,r,l} = 0$, then the system $\{y_k(x)\}_{k=1,\, k \neq i,j,r,l}^{\infty}$ is not complete and not minimal in $L_p(0,1)$, $1 < p < \infty$.*

Using Theorem 2 we can establish sufficient conditions for the $\{y_k(x)\}_{k=1,\, k \neq i,j,r,l}^{\infty}$ of eigenfunctions of problem (1)-(5) to form a basis in the space $L_p(0,1)$, $1 < p < \infty$.

We assume that $y_1(x) = 1$ and $y_2(x) = 1 - x$, $x \in [0,1]$.

Theorem 3. *Let $i = 1$, $j = 2$ and r, l, $r < l$, be arbitrary sufficiently large fixed natural numbers which are even, and let $c \neq |a|$. Then the system $\{y_k(x)\}_{k=1,\, k \neq i,j,r,l}^{\infty}$ is a basis in $L_p(0,1), 1 < p < \infty$, and for $p = 2$ this basis is an unconditional basis.*

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Bessel's inequality and basis property for a Dirac type system $2m \times 2m$

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We consider a Dirac type $2m$-th order operator:

$$Du = B\frac{du}{dx} + \Omega\ (x)\ \ u, u = (u_1, u_2, ..., u_{2m})^T, m \geq 1$$

Determined on an arbitrary interval $G = (a, b)$, where $B = \begin{pmatrix} 0 & J \\ -J & 0 \end{pmatrix}$ or $B = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$, I is a unit operator in E^m, $J_m = (\alpha_{ij})_{i,j=1}^m$, $J_m = (\alpha_{ij})_{i,j=1}^m$, $\alpha_{k,\ m-k+1} = 1$, $k = \overline{1, m}$; $\alpha_{ij} = 0$, $(i, j) \neq (k,\ m - k + 1)$, $k = \overline{1, m}$; $\Omega\,(x)$ is a summable complex valued $2m \times 2m$ matrix function.

The following theorems are proved in this work.

Theorem 1. *(Bessel property criterion). Let $\Omega\,(x) \in L_1\,(G) \otimes C^{2m\,\times\,2m}$, the length of the chains of the root vector-functions be uniformly bounded, and there exists such a constantC_0that*

$$|Im\,\lambda_k| \leq C_0,\ \ k = 1, 2, ... \tag{1}$$

For the system $\left\{\psi_k\,(x)\ \|\psi_k\|\ {}_2^{-1}\right\}_{k=1}^{\infty}$ to be Bessel in $L_2^{2m}\,(G)$, it is necessary and sufficient that there exists such a constant M_1 that

$$\sum_{|Re\,\lambda_k - \tau| \leq 1} 1\ \ \leq M_1, \tag{2}$$

where τ is an arbitrary real number.

Denote by D^* an operator formally adjoint to the operator D, i.e.

$$D^* = B\frac{d}{dx} + \Omega^*\ (x)\,,$$

where $\Omega^*\ (x)$ is a matrix ad joint to the matrix $\Omega\ (x)$.

Let the system $\{\psi_k(x)\}_{k=1}^{\infty}$ be minimal $L_2^{2m}(G)$, and its biorthogonally conjugate system $\{\varphi_k(x)\}_{k=1}^{\infty}$ consist of the root vector-functions of the operator D^*.

Theorem 2. *(On unconditional basicity). Let $\Omega(x) \in L_1(G) \otimes C^{2m \times 2m}$, the lengths of the chains of the root vector-functions be uniformly bounded, one of the systems $\{\psi_k(x)\}_{k=1}^{\infty}$ and $\{\varphi_k(x)\}_{k=1}^{\infty}$ be complete in $L_2^{2m}(G)$ and condition (1) be fulfilled.*

Then the necessary and sufficient condition funconditional basicity in $L_2^{2m}(G)$ of each of these systems is the existence of the constants M_1 and M_2, providing the validity of inequalities (2) and

$$\|\psi_k\|_2 \|\varphi_k\|_2 \le M_2, k = 1, 2, ... \tag{3}$$

Theorem 3. *(On equivalent basicity). Let $\Omega(x) \in L_1(G) \otimes C^{2m \times 2m}$, conditions (1)-(3) be fulfilled and the system $\left\{\psi_k(x) \|\psi_k\|_2^{-1}\right\}_{k=1}^{\infty}$ be quadratically close to some basis $\{u_k(x)\}_{k=1}^{\infty}$ of the space $L_2^{2m}(G)$.*

Then the systems $\left\{\psi_k(x) \|\psi_k\|_2^{-1}\right\}_{k=1}^{\infty}$ and $\{\varphi_k(x) \|\psi_k\|_2\}_{k=1}^{\infty}$ are the bases in $L_2^{2m}(G)$ are equivalent to the basis $\{u_k(x)\}_{k=1}^{\infty}$ and its biorthogonal conjugate system, respectively.

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On the strong solvability of a nonlocal boundary value problem for the Poisson's equation in a rectangular

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Consider the following nonlocal boundary value problem for the Poisson's equation:

$$u_{xx} + u_{yy} = f\left(x, y\right), \quad 0 < x < 2p, 0 < y < h, \tag{1}$$

$$u\left(x, 0\right) = \varphi\left(x\right), u\left(x, h\right) = \psi\left(x\right), \qquad 0 < x < 2\pi, \tag{2}$$

$$u_x\left(0, y\right) = 0, u\left(0, y\right) = u\left(2\pi, y\right), \ 0 < y < h. \tag{3}$$

Such problems have specific features in comparison with problems with local conditions. Earlier F. I. Frankl [1; 2, p. 453-456] considered a problem with a non-local boundary condition for a shifted type equation. In works [3,4], problems (1)-(3) are considered in an infinite strip in the classical formulation. It was considered a nonlocal boundary value problem for the Laplace equation in an unbounded domain studying the weak and strong solvability of that problem in the framework of the weighted Sobolev space in the works of B.T.Bilalov [5]. The solution of the problem (1)-(3) is sought as the sum of the solution of two cases as follows. (A) $\{f\left(x; y\right) = 0, \varphi\left(x\right) \neq 0 \ and \ \psi\left(x\right) \neq 0\}$ and (B) $\{f\left(x; y\right) \neq 0 \ and \ \varphi\left(x\right) = \psi\left(x\right) = 0\}$. In the case of (A) this equation was considered in [6] in a bounded domain in weighted Sobolev spaces. In this work, we will look at the case with of (B). In this paper, we consider problems (1)-(3) in a weighted Sobolev space with a weight from the Mackenhoupt class. Let $\nu : [0, 2\pi] \to (0, +\infty)$ be some weight function, $\Pi = (0, 2\pi) \times (0, h)$, $I = (0, 2\pi)$.

Definition 1. *A function $u \in W^2_{p,\nu}\left(\Pi\right)$ is called a strong solution to problem (1)-(3) if equality (1) is satisfied a.e. $(x; y) \in \Pi$, and its trace $u \mid_{\partial\Pi}$ satisfies relations (2), (3).*

Let us introduce into consideration the system of functions $\{u_n\left(x\right)\}_{n\in Z^+}$ and $\{\vartheta_n\left(x\right)\}_{n\in Z^+}$, where

$$u_{2n}\left(x\right) = cos \ nx, n \in Z^+, u_{2n-1}\left(x\right) = xsin \ nx, n \in N, \tag{4}$$

$$\vartheta_0(x)=\frac{1}{2\pi^2}(2\pi-x),\vartheta_{2n}(x)=\frac{1}{\pi^2}(2\pi-x)\,cos\,nx,\vartheta_{2n-1}(x)=\frac{1}{\pi^2}sin\,nx,n\in N. \tag{5}$$

Note that systems (4) and (5) are biorthogonally conjugate, which can be verified directly. In obtaining the main result, the following theorem is essentially used.

Theorem 1. *Let $\nu\in A_p(I)$, $1<p<+\infty$. Then system (4) forms a basis in $L_{p;\nu}(I)$.*

The solution to problem (1)-(3) is sought in the form of a series $u(x,y)=\sum_{n=0}^{\infty}U_n(y)\,u_n(x)$, where

$$U_0(y)=-\frac{y}{h}\int_0^h(h-t)\,F_0(t)\,dt+\int_0^y(y-t)\,F_0(t)\,dt,$$

$$U_{2n-1}(y)=-\frac{1}{n}\frac{\sinh n\,(h-y)}{\sinh nh}\int_0^y\sinh nt\;F_{2n-1}(t)\,dt-$$

$$-\frac{1}{n}\frac{\sinh ny}{\sinh nh}\int_y^h\sinh n(h-t\;)F_{2n-1}(t)\,,$$

$$U_{2n}(y)=-\frac{1}{n}\frac{\sinh n\,(h-y)}{sinhn\;h}\int_0^yF_{2n}(t)\sinh nt\;dt-\frac{1}{n}\frac{\sinh ny}{\sinh nh}\int_0^yF_{2n}(t)\sinh nt\;dt-$$

$$-\frac{2}{n}\frac{\sinh n\,(h-y)}{\sinh nh}-\frac{2}{n}\frac{\sinh n\,(h-y)}{\sinh nh}\int_0^y\frac{\sinh nt}{\sinh nh}\int_t^hF_{2n-1}(\tau)\sinh n\,(h-\tau)\;\,d\tau dt\;-$$

$$-\frac{2}{n}\frac{\sinh ny}{\sinh nh}\int_y^h\frac{\sinh n(h-t\;)}{\sinh nh}\int_0^tF_{2n-1}(\tau)\sinh n\tau\;\,d\tau dt\;\;-$$

$$-\frac{2}{n}\frac{\sinh ny}{\sinh nh}\int_y^h\frac{\sinh nt}{\sinh nh}\int_t^hF_{2n-1}(\tau)\sinh n\,(h-\tau)\,d\tau dt,\quad n\in N,$$

$$F_n(y)=\int_0^{2\pi}f(x,y)\,\vartheta_n(x)\,dx,n\in Z^+.$$

The main result of the paper is the following theorem:

Theorem 2. *Let $\nu\in A_p(I)$, $1<p<+\infty$, the boundary functions $\varphi(x)=\psi(x)=0$ and $f(x,y)\in W_{p,\nu}^1(\Pi)$ and satisfies the following condition*

$$f(0,y)=f(2\pi,y)\,.$$

Then problem (1)-(3) has a unique solution in $W^2_{p,\nu}(\Pi)$ *and moreover, it is valid the following estimate*

$$\|u\|_{W^2_{p;\nu}(\Pi)} \leq c\|f\|_{W^1_{p,\nu}(\Pi)},$$

where $c > 0$ *is a constant independent of* $f(x, y)$.

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Uniform convergence of spectral expansion in eigenfunctions of a fourth-order differential operator

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On the interval $G = (0, 1)$, consider the differential operator

$$Lu = u^{(4)} + P_2(x)u^{(2)} + P_3(x)u^{(1)}\, P_4(x)u$$

with coefficients $P_l(x) \in L_1(G),\ \ l = \overline{2,4}$.

By $D_4\,(G)$ we denote the class of functions absolutely continuous together with their derivatives of order ≤ 3 on the interval $\overline{G} = [0, 1]$.

Under the $\{u_k(x)\}_{k=1}^{\infty}-$eigenfunction of the operator L corresponding to the eigenvalue λ we will understant any identically non-zero function $y(x) \in D_4(G)$, satisfying almost everywhere in G the equation $Ly + \lambda y = 0$ (see [1]).

Let $\{u_k(x)\}_{k=1}^{\infty}-$be a complete orthonormed in $L_2\,(G)$ system consisting of eigenfunctions of the operator L, and let while $\{\lambda_k\}_{k=1}^{\infty}$ be the corresponding system of eigenvalues, $\lambda_k \leq 0$.

By $W_p^1(G)$, $p \geq 1$, we denote the class of functions $f(x)$ absolutely continuous on the interval $\overline{G}$ for which $f'\,(x) \in L_{\,p}(G)$. We write $\mu_k = ((-1)^{m+1}\lambda_k)^{1/2m}$, and introduce a partial sum of the orthogonal expansion of the function $f(x) \in W_1^1(G)$ with respect to the system $\{u_k(x)\}_{k=1}^{\infty}$:

$$\sigma_\nu(x, f) = \sum_{\mu_k \leq \nu} f_k u_k(x)\,,\ \nu > 0,$$

where $f_k = (f, u_k) = \int_0^1 f(x)\overline{u_k(x)}dx$.

In this work is proven following theorem.

Theorem. *Let the function* $f(x) \in W_p^1(G),\ p > 1$.*Then*

1. *For uniform convergence of the series*

$$\sum_{k=1}^{\infty} |f_k|\; |u_k(x)|\ ,\quad x \in \overline{G}, \tag{1}$$

necessary and sufficiently uniform convergence of the series

$$\sum_{\mu_k \geq 1}^{\infty} \mu_k^{-4} \; |\alpha_k\,(f)| \; |u_k(x)| \;, \quad x \in \overline{G}, \tag{2}$$

where $\alpha_k\,(f) = f\,(1)\,\overline{u_k^3\,(1)} - f\,(0)\,\overline{u_k^3\,(0)}$.

2. *For uniform convergence of the series*

$$\sum_{k=1}^{\infty} f_k u_k(x), \qquad x \in \overline{G}, \tag{3}$$

necessary and sufficiently uniform convergence of the series

$$\sum_{\mu_k \geq 1}^{\infty} \mu_k^{-4} \alpha_k\,(f)\; u_k(x), \qquad x \in \overline{G}. \tag{4}$$

3. *If the function $f(x)$ and system $\{u_k(x)\}_{k=1}^{\infty}$ satisfy the condition*

$$\sum_{\mu_k \geq 1}^{\infty} \mu_k^{-4} \alpha_k u_k(x)\,(f)\,, \quad x \in \overline{G}, \tag{5}$$

then spectral expansion (3) of the function $f\,(x)$ in the system $\{u_k(x)\}_{k=1}^{\infty}$ absolutely and uniformly converges on $\overline{G} = [0,1]$ and for the remainder $R_\nu(x,f) = f(x) - \sigma_\nu(x,f)$ the following estimations are valid

$$\|R_\nu\,(\cdot,f)\|_{C\,[0,1]} \leq const \left\{\, C_1\,(f)\,\nu^{-1} + \leq \right.$$

$$\left. +\nu^{-\beta}\,\|f'\|_p + \nu^{-1}\left(\|f\|_\infty + \|f'\|_p\right) \sum_{l=2}^{4} \nu^{2-l}\,\|P_l\|_1 \right\} \tag{6}$$

$$\|\, R_\nu\,(\,\cdot\,,\,f)\,\|_{\;C\,[0,1\,]} = o\,\left(\nu^{-\beta}\right), \tag{7}$$

where $\nu \geq 1$, $p^{-1} + q^{-1}$, $\beta = min\{2^{-1}, q^{-1}\}$, $\|\cdot\|_p = \|\cdot\|_{L_p(G)}$; the const is independent of $f(x)$ and symbol "o" is dependent of $f(x)$.

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Wiener-type criterion for the heat equation in terms of the parabolic potential and its corollaries

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In a bounded domain $D \subset \mathbb{R}^{n+1}$ we consider the Dirichlet problem for the heat equation

$$\triangle\, u - u_t = 0, \quad u|_{\partial_p D} = f \in C(\partial_p D)$$

and we get the Wiener type criterion in potential terms for the regularity of the boundary point $(x_0, t_0) \in \partial D$ with respect to this problem.

Definition. A point $(x_0, t_0) \in \partial D$ is said to be regular if

$$\lim_{D \ni (x,t) \to (x_0,t_0)} u_f\,(x,t) = f\,(x_0, t_0)$$

for any $f \in C\,(\partial_p D)$, where $u_f\,(x,t)$ is the generalized solution in the Wiener sense for the problem (1).

It should be noted that the Wiener-type regularity criterion in terms of the heat capacity of the boundary point for the heat equation was proved by Evans and Gariepy [1]. Their method is based on the mean value theorem and can cover only the class of parabolic equations with divergent structures

[2]. Our method can also cover the class of parabolic equations with a non-divergent structure. Let's introduce some necessary concepts and notations. For simplicity of notation, we will henceforth assume $(x_0, t_0) = (0, 0) \in \partial D$.

Let $B \subset R^{n+1}$, $(x,t) = (\ x_1, \ldots, x_n, t)$ and

$$P_B\,(x,t) = \int_B K(t-\tau, x-\xi)d\mu(\tau,\xi)$$

be heat potential with generated of Weierstrass kernel

$$K\,(t,x) = \left\{ \begin{array}{l} (4\pi t)^{-\frac{n}{2}} \\ 0,\ t \le 0 \end{array} \right. \cdot exp \left\{ -\frac{|x|^2}{4t} \right\},\ t > 0,$$

here B is a Borel set and μ is a Borel measure. Denote for $\lambda > 1$ and $m \in N \cup \{0\}$ paraboloids: $P_m = \left\{(t,x) \mid |x|^2 < -\lambda^m \cdot t, t < 0\right\}$.

Let $B_{m,k} = (P_{m+1} \backslash P_m) \cap \left[-t_k; -\frac{3t_k}{4}\right]$, where $t_{k+1} = \frac{t_k}{4}$, $k \in N \cup \{0\}$, $t_0 > 0$, and denote cylinders

$C_{m,k} = \{(t,x) \mid -t_k < t < 0,\ |x| < a \cdot \rho_{m,k}\}$, where $a > 0$ the chosen absolute constant depending only on the dimension n of the space and denote by $S_{m,k}$ the lateral surface of the cylinder $C_{m,k}$. The measure μ on B is called admissible, if $P_B\,(t,x) = \int_B K\,(t-\tau, x-\xi)\,d\mu(\tau,\xi) \le 1$ in R^{n+1}.The number $cap\,(B) = supB$, where the supremum is taken by all possible admissible measures is called the thermal capacity of the set B.

Let's call $T_{m,k} = C_{m,k} \backslash P_m$ trapezoids and denote by $T_{m,k}^{(j)}, j = 1, 2, \ldots, n_0(n)$ corresponding minimal finite partition $T_{m,k}$, for which the following $|x - \xi| \le |\xi|$ inequality is fulfilled at $(x,t) \in T_{m,k+1}^{(j)}$ and $(\xi, t) \in T_{m,k}^{(j)}$ for every fixed $j \in \{1, \ldots, n_0\}$.

Lemma 1. If $(x,t) \in T_{m,k+1}^{(j)}$ and $(\xi,t) \in T_{m,k}^{(j)}$, then

$$\frac{|x-\xi|^2}{t-t} \le \frac{|\xi|^2}{-\tau}\ . \tag{2}$$

Lemma 2. There exist the following absolute constants $C_1 > 0$ and $C_2 > 0$, depending only on fixed numbers λ, a and n such that holds

$$\sup_{S_{m,k}} P_{T_{m,k}^{(j)} \cap B_{m,k}}\,(t,x) \le C_1.\ \cdot P_{T_{m,k}^{(j)} \cap B_{m,k}}\,(0,0) \tag{3}$$

and also such finite partition that at every $j \in \{1, \ldots, n_0\}$

$$\min_{T^{(J)}_{M,k+1}} P_{T^{(j)}_{m,k}\cap B_{m,k}}(t,x) \ \geq C_2 \cdot P_{T^{(j)}_{m,k}\cap B_{m,k}}(0,0)\,, \text{ moreover } C_2 > C_1. \qquad (4)$$

Lemma 3. Let bounded domain $D \subset R^{n+1}$ containing in cylinder $C_{m,k}$, intersecting by cylinder $C_{m,k+1}$: $D \cap C_{m,k+1} \neq \emptyset$ and $u(t,x)$ be a solution of the equation (1) positive in D, continuous in $\overline{D}$ and vanishing on such part of the parabolic boundary $\partial_p D$ of the domain D, which lies strongly inside $C_{m,k}$.Then for every $j \in \{1,\ldots,n_0\}$ we have

$$\sup_{D\cap T^{(j)}_{m,k}} u(t,x) \ \geq \left(1+\eta P_{D^c\cap T^{(j)}_{m,k}\cap B_{m,k}}(0,0)\right) \sup_{D^c\cap T^{(j)}_{m,k+1}} u(t\ ,x), \qquad (5)$$

where $\eta > 0$- absolute constant and $D^c = \mathbb{R}^{n+1}\backslash D$. Now we formulate the main result of this thesis.

Theorem 1. A point $(\mathbf{0},0) \in \partial D$ is regular with respect to the Dirichlet problem (1) if and only if

$$\sum_{m,k=1}^{\infty} P_{B_{m,k}}(0,0) = \infty.$$

Corollary 1. [1] A point $(0,0) \in \partial D$ is regular with respect to the Dirichlet problem (1) iff

$$\sum_{m=1}^{\infty} e^{mn/2} cap\left(D^c \cap A\left(e^{-m}\right)\right) = \infty.$$

Corollary 2. For the symmetrical domains

$$D_H = \left\{(x,t) : -H < t < 0, |x|^2 < 4t \log\rho(t)\right\} \subset \Re^{n+1}, H > 0.$$

a generalization of Petrovsky's [3] result is obtained:

Theorem 2. *Let $\rho(t) > 0$ be a monotony decreasing function vanishing at the origin as $t \to -0$. Assume that, $t\rho(t) \to 0$as $t \to -0$.*

Then the point $(0,0) \in \partial D_H$is regular for the problem (1),(2) if and only if

$$\lim_{\varepsilon\to 0} \int_H^{\varepsilon} \frac{\rho(\eta)\left|\log\rho(\eta)\right|^{n/2}}{\eta} d\eta = -\infty. \qquad (13)$$

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Optimal control problems for a system of second order hyperbolic equations

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Abstract. In the paper, the inverse problem for a system of second-order hyperbolic equation with additional conditions is reduced to an optimal control problem. A theorem on the existence of optimal control is proved and the necessary condition of optimality is derived in the form of a variational inequality.

Keywords: *system of hyperbolic equation, inverse problem, optimal control problem, optimality condition.*

Let the process be described by the system of differential equations

$$\frac{\partial^2 u}{\partial t^2} - A\frac{\partial^2 u}{\partial x^2} = f(x,t) \tag{1}$$

In the the domain $Q = (0;l) \times (0;T)$, where $u = [u_1(x,t), u_2(x,t)]'$ is a vector-function, A is a constant, positive-definite diagonal matrix of second order, $f(x,t) \in (L_2(Q))^2$ is a vector-function.

Let the initial conditions

$$u(x,0)=\phi_0(x),\ \frac{\partial u(x,0)}{\partial t}=\upsilon(x),\ 0\le x\le l, \tag{2}$$

and the boundary conditions

$$u(0,t)=0,\ u(l,t)=0,\ 0\le t\le T, \tag{3}$$

are satisfied, where $l,\ T$- positive number, $\phi_0(x)\in\left(\overset{o}{W}{}_2^1[0,l]\right)^2$ is the given vector functions.

If the function $\upsilon(x)\in(L_2[0,l])^2$ is given, it is easily proved by analogy to the work [5] that the problem (2.1)-(2.3) has a unique generalized solution from $\left(W_{2,0}^1(Q)\right)^2$.

If $\upsilon(x)\in(L_2[0,l])^2$ is an unknown vector-function, then for determine $\upsilon(x)$ an additional condition is specified, for example, in the form

$$\int_0^T \langle K(x,t),u(x,t)\rangle\,\mathrm{dt}=\chi(x),\ 0\le x\le l, \tag{4}$$

where $\chi(x)\in L_2[0,l]$, - is given function, $K(x,t)\in(L_\infty(Q))^2$ - is the given vector-function, $\langle\cdot,\cdot\rangle$- scalar product of vectors from R_2.

We reduce this problem to the following optimal control problem: find a function $\upsilon(x)$ that minimizes the functional

$$J(\upsilon)=\frac{1}{2}\int_0^l\Big|\int_0^T\langle K(x,t),u(x,t;\upsilon)dt-\chi(x)\rangle\Big|^2+\alpha\int_0^l|\upsilon(x)|_{R^2}^2\,dx\to\min, \tag{5}$$

together with the solution of the boundary value problem (1)-(3), here $\alpha>0$ constant.

There is a close connection between problems (1)-(4) and (1)-(3), (5): if $\min\limits_{\upsilon\in V}J(\upsilon)=0$, then the additional condition (4) is satisfied.

The work proves the existence theorem for optimal control and derives the necessary condition for optimality in the form of a variational inequality.

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An optimal control problem for a wave equation with the third nonlocal boundary condition

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A nonlocal problem for hyperbolic equations has been considered in [2]. An optimal control problem for such equations has been considered for example in [3,4].

Let a controlled process in the domain $Q = \Omega \times (0, T)$ be described by the wave equation

$$\frac{\partial^2 u}{\partial t^2} - \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} = v\,(x, t)\,, \;\; x \in Q \tag{1}$$

with initial condition

$$u\,(x, 0) = u_0\,(x)\,, \;\; \frac{\partial u(x, 0)}{\partial t} = u_1\,(x)\,, x \in \Omega \tag{2}$$

and the boundary condition

$$\left(\frac{\partial u}{\partial \nu} + \sigma(x, t)u \right)\bigg|_S = \int_\Omega K\,(x, y)\, u\,(y, t)\, dy, \;\; (x, t) \in S, \tag{3}$$

where Ω is a bounded domain in R^n with a smooth boundary $\partial\Omega$, $\nu\,(x, t)$ is an admissible control from the convex, closed set $U_m \subset L_2(Q)$, $u_0\,(x) \in W_2^1\,(\Omega)$, $u_1\,(x) \in L_2(\Omega)$, ν is an exernal normal to S, $K\,(x, y) \in L_\infty(\Omega \times \Omega,)$, $K(x, y) = K\,(y, x)$, $\int_\Omega \int_\Omega \left(K^2\,(x, y) + \left| \frac{\partial K(x,y)}{\partial x} \right|^2 \right) dxdy < \infty$ is the given functions, $\sigma(x, t)$ be a given function bounded with the first derivative with respect to t.

Here $u = u\,(x, t) = u\,(x, t; v)$ is a generalized solution of the problem (1)-(3) from the space $W_2^1\,(Q)$ [1], corresponding to the control $v\,(x, t)$.

Note that under the generalized solution of the problem (1)-(3) we mean such a function from $W_2^1\,(Q)$, for which the following integral identity holds :

$$\int_0^T \int_\Omega \left(-\frac{\partial u}{\partial t}\frac{\partial \eta}{\partial t} + \sum_{i=1}^{n} \frac{\partial u}{\partial x_i}\frac{\partial \eta}{\partial x_i} \right) dxdt -$$

$$- \int_0^T \int_\Omega \eta(x, t) \int_\Omega K\,(x, y)\, u\,(y, t)\, dydsdt + \int_0^T \int_\Omega \sigma(x, t)\, u\,(x, t)\, \eta\,(x, t)\, dsdt -$$

$$- \int_\Omega u_1(x)\eta\,(x, 0)\, dx = \int_0^T \int_\Omega v(x, t)\eta\,(x, t)\, dxdt, \tag{4}$$

$\forall \eta \in W_2^1(Q), \eta(x, T) = 0$.

The following problem is considered: among all admissible controls, find the one which together with the solution of the initial boundary value problem

(1)-(3) gives a minimum to the functional

$$J(v) = \frac{1}{2}\int_0^T \int_\Omega (u - \overline{u})^2 dxdt + \frac{N}{2}\int_0^T \int_\Omega (v(x,t) - \omega(x,t))^2 dxdt, \qquad (5)$$

where $\overline{u}(x,t)$ and $\omega(x,t)$ are the given functions from $L_2(Q)$, $N > 0$ is a given number .

In the paper we prove the existence of optimal control and derive a necessary and sufficient condition for optimality in the form of a variational inequality:

$$\int_0^T \int_\Omega (\Psi_*(x,t) + N(v_*(x,t) - \omega(x,t))(v(x,t) - (v_*(x,t))\, dxdt \geq 0 \ \forall v \in U_m,$$

where $\Psi_*(x,t)$ is the solution to the following adjoin problem, corresponding to admissible control $v_*(x,t)$:

$$\frac{\partial^2 \Psi}{\partial t^2} - \sum_{i=1}^{n} \frac{\partial^2 \Psi}{\partial x_i^2} = -(u - \overline{u}) + \int_{\partial\Omega} K(\xi, x)\, \Psi(\xi, t)\, ds, \ (x,t) \in Q,$$

$$\Psi(x,T) = 0, \ \frac{\partial \Psi(x,T)}{\partial t} =, x \in \Omega \ ,$$

$$\left(\frac{\partial \Psi}{\partial \nu} + \sigma(x,t)\Psi \right)\Big|_S = 0, \ (x,t) \in S.$$

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Weak solvability of the Dirichlet problem of the completed form Gilbarg-Serrin equation

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In this thesis involve the Dirichlet problem for the completed form Gilbarg-Serrin equation is considered. The unique weak solvability of this problem in corresponding weighted Sobolev space is proved.

Let D be a bounded domain, placed in n-dimensional Euclidean space E_n of points $x = (x_1, x_2, ..., x_n, n \geq 3, 0 \in D)$ with boundary ∂D and consider Gilbarg-Serrin operation in D:

$L = \Delta + \mu(r) + \sum_{i,j=1}^{n} \frac{x_i x_j}{r_2} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial}{\partial x_i} + c(x)u(x)$ where $r = |x|$, $b_1 \leq \mu(r) \leq \ b_2, b_1 > -1, b_2 < \infty . b_i(X) \in \ C^2(D) \, (i = \bar{1, n}), C(x) \in \ C(D)$ Now we will introduce some preliminaries and definitions.By $L_{2,\gamma}(D)$ and $W_{2,\gamma}^1(D)$ we will denote Banach spaces of functions, defined on D, with finite norms $\|u\|_{2,\gamma(D)} = \left(\int_D r^{\gamma-2} u^2 dx\right)^{\frac{1}{2}}$ and $\|u\|_{W_{2,\gamma(D)}^1} = \left(\int_D (r^{\gamma-2} u^2 + r^\gamma |\nabla u|^2) dx\right)^{\frac{1}{2}}$ correspondingly.Here ∇u is a gradient vector of the function $u(x)$.By $\dot{W}_{2,\gamma}^1(D)$ we denote subspace of $W_{2,\gamma}^1(D)$, the dense set of which is the aggregate of all infinitely differentiable functions with compact support in D, with finite corresponding norm. Let for $i, j = 1, \ldots, n$, $u_i = \frac{\partial u}{\partial x_i}$, $u_{ij} = \frac{\partial^2 u}{\partial x_i \partial x_j}$. Further, the record C means that positive constant and it depends only on parameters in brackets. For function $\mu(r)$ we suppose the following conditions $b_0 \leq \mu(r) \leq \ b_2,\ b_0 > 2n-3,\ b_2 < \infty\ ; |\mu'(r)| \leq \frac{C_1}{r};\ -\frac{C_2}{r^2} \leq \mu^n(r) \leq 0;$ $r \in (0, diamD)$.

Lemma 1. *Let in bounded domain $D \subset E_n$ the coefficients of operator L be defined. If $\gamma = 1 - n$, then for any function $u(x) \in \dot{W}_{2,\gamma}^1(D)$ holds the following inequality:* $-\int_D r^\gamma u L u dx \geq C_3(\mu) \int_D r^\gamma |\nabla u|^2 dx$

Lemma is proved for $C_3=\delta$.

Now we will introduce bilinear form for $u,\vartheta\in \dot{W}^1_{2,\gamma}(D)$,

$B(u,\vartheta)=\sum_{i=1}^n r^\gamma u_i\vartheta_i dx+\gamma\int_D r^{\gamma-2}u\vartheta dx-\gamma\sum_{i=1}^n\int_D r^{\gamma-2}x_i\vartheta_i u dx+$

$$+\sum_{i.j=1}^n\int_D r^{\gamma-2}x_ix_j\mu(r)\,u_i\vartheta_j dx+\int_D r^{\gamma-1}\mu'(r)\,u\vartheta dx-\int_D r^\gamma\mu''(r)\,u\vartheta dx-$$

$$-\sum_{i=1}^n\int_D r^{\gamma-1}\mu'(r)\,x_iu\vartheta_i dx.$$

Lemma 2. *Bilinear form $B(u,\vartheta)$ is bounded and coercive in $\dot{W}^1_{2,\gamma}(D)$, i.e. there exist constants $C_4(\mu,n)$ and $C_5(\mu,n)$ such that for any $u,\vartheta\in\dot{W}^1_{2,\gamma}(D)$ $|B(u,\vartheta)|\leq C_4\|u\|\,\|\vartheta\|$, $B(u,\vartheta)\geq C_5\|u\|^2$ where $\|\cdot\|=\|\cdot\|_{W^1_{2,\gamma}(D)}$.*

Lemma is proved for $C_5=\frac{C_3}{8}$.

We will call the weak solution of Dirichlet problem $Lu=g+\sum_{i=1}^n\frac{\partial f^i}{dx_i}$, $u|_{\partial D}=0$, where $g\in L_{2,\gamma+2}(D)$, $f^i\in L_{2,\gamma}(D)$, $i=1,\ldots,n$; the function $u(x)\in\dot{W}^1_{2,\gamma}(D)$ such that for any function $\vartheta(x)\in\dot{W}^1_{2,\gamma}(D)$ holds the following integral identity. $B(u,\vartheta)=-\int_D r^\gamma g\vartheta dx+\sum_{i=1}^n\int_D r^\gamma\vartheta_i f^i dx+\gamma\sum_{i=1}^n\int_D r^{\gamma-2}x_i\vartheta f^i dx$.

Theorem 1. *The Dirichlet problem has a unique weak solution $u(x)\in\dot{W}^1_{2,\gamma}(D)$ for all $g\in L_{2,\gamma+2}(D)$, $f^i\in L_{2,\gamma}(D)$, $i=1,\ldots,n$.*

Theorem 2. *For the weak solution $u(x)$ of Dirichlet problem the following estimate holds $\|u\|_{W^1_{2,\gamma}(D)}\leq C_6(\mu,n)\left(\|g\|_{L_{2,\gamma+2}(D)}+\sum_{i=1}^n\|f^i\|_{L_{2,\gamma}(D)}\right)$.*

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Global bifurcation from zero and infinity in nonlinear Sturm-Liouville problems with indefinite weight and spectral parameter in the boundary conditions

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Consider the following nonlinear differential equation

$$\ell(y) \equiv -(p\,(x)y')' + q(x)y = \lambda r(x)y + f(x, y, y, \lambda), x \in (0, 1), \tag{1}$$

subject the boundary conditions

$$b_0 y(0) = d_0 p(0) y'(0), \tag{2}$$

$$(a_1\lambda + b_1)y(1) = d_1 p(1) y'(1), \tag{3}$$

where $\lambda \in \mathbb{R}$ is a parameter, p is a positive and continuously differentiable function on $[0, 1]$, q is a nonnegative continuous function on $[0, 1]$, r is a sign-chaning continuous function on $[0, 1]$, b_0, d_0, a_1, b_1, d_1 are real constants such that $|b_0| + |d_0| > 0$, $b_0 d_0 \geq 0$, and if $b_0 = 0$, then q is not identically zero, and $a_1 d_1 > 0$, $b_1 d_1 \leq 0$. The function f is continuous on $[0, 1] \times \mathbb{R}^3$ and satisfies the following conditions:

$$uf(x, u, s, \lambda) \leq 0, x \in [0, 1], (u, s) \in R^2, \lambda \in \mathbb{R}; \tag{4}$$

there exist a positive number M, a positive sufficiently small number τ_0 and a positive sufficiently large number τ_1 such that

$$\begin{gathered} \left|\frac{f(x,u,s,\lambda}{u}\right| \leq M, x \in [0, 1], (u, s) \in R^2, u \neq 0, |u| + |s| \leq \tau_0, \\ |u| + |s| \geq \tau_1, \lambda \in \mathbb{R}. \end{gathered} \tag{5}$$

In this note, we study the global bifurcation of nontrivial solutions to problems (1)-(3) from zero and infinity.

As is known (see [1]), problem (1)-(3) describes selection-migration in population genetics.

The eigenvalues of the linear eigenvalue problem

$$\begin{cases} \ell(y) = \lambda r(x) y, x \in (0,1), \\ b_0 y(0) = d_0 p(0) y'(0),\ (a_1\lambda + b_1) y(1) = d_1 p\,(1) y'(1), \end{cases} \tag{6}$$

are real, simple and form two infinitely increasing and infinitely decreasing sequences

$$\ldots < \lambda_k^- < \ldots < \lambda_2^- < \lambda_1^- < 0 \text{ and } 0 < \lambda_1^+ < \lambda_2^+ < \ldots < \lambda_k^+ < \ldots ,$$

respectively; for each $k \in \mathbb{N}$ the eigenfunctions y_k^+ and y_k^- corresponding to the eigenvalues λ_k^+ and λ_k^-, respectively, have exactly $k-1$ simple zeros in the interval $(0,1)$ (see [2]).

Let $E = C^1[0,1] \cap \{y : b_0 y(0) = d_0 p(0) y'(0)\}$ be the Banach space with the usual norm $||u||_1 = \max\limits_{x\in[0,1]} |u(x)| + \max\limits_{x\in[0,1]} |u(x)|$.

From now on σ and ν will denote either $+$ or $-$; $-\sigma$ and $-\nu$ will denote the opposite sign to σ and ν.

For each λ, each $k \in \mathbb{N}$, each σ and each ν by $S_{k,\lambda}^{\sigma,\nu}$ we denote the set of functions $y \in E$ which satisfy the conditions:

(i) $(a_1\lambda + b_1) y(1) = d_1 p\,(1) y'(1)$;

(ii) the function y has exactly $k-1$ simple zeros in $(0,1)$;

(iii) $\sigma \left\{ \int\limits_0^1 r(x) y^2(x) dx + \frac{a_1}{d_1} y^2(1) \right\} > 0$;

(iv) $\lim\limits_{x\to 0+} \nu y(x) = 1$.

For each $k \in \mathbb{N}$, each σ and each ν, let $S_k^{\sigma,\nu}$ be the set defined by

$$S_k^{\sigma,\nu} = \bigcup_{\lambda\in R} S_{k,\lambda}^{\sigma,\nu}.$$

The sets $S_k^{\sigma,\nu}$ are disjoint open subsets of E; moreover, if $y \in \partial S_k^{\sigma,\nu}$, then, either (i) there exists $\zeta \in [0,1]$ such that $y(\zeta) = y'(\zeta) = 0$, or (ii) $\int\limits_0^1 r(x) u^2(x) dx + \frac{a_1}{d_1} u^2(1) = 0$ [3].

Lemma 1. *Let (λ, y) be a solution to problem (1)-(3) such that $y \in \partial S_k^{\sigma,\nu}$. Then $y \equiv 0$.*

Let λ_1 be the smallest eigenvalue of linear problem (6) with $r \equiv 1$. It follows from [2, Lemma 2.1] that λ_1 is positive.

Lemma 2. *For each $k \in \mathbb{N}$, each σ and each ν the set of bifurcation points from zero (respectively, from infinity) of problem (1)-(3) with respect to the set $\mathbb{R} \times S_k^{\sigma,\nu}$ is non-empty. Moreover, if (λ, ∞) (respectively, $(\lambda, 0)$) is such a point, then $\lambda \in I_k^{\nu}$, where*

$$I_k^+ = [\lambda_k^+, \lambda_k^+ + d_k^+],\ I_k^- = [\lambda_k^- - d_k^-, \lambda_k^-], d_k^+ = \frac{M\lambda_k^+}{\lambda_1},\ d_k^- = -\frac{M\lambda_k^-}{\lambda_1}.$$

Let D be the set of nontrivial solutions of problem (1)-(3). For each $k \in \mathbb{N}$, each σ and each ν by $D_k^{\sigma,\nu}$ (respectively, $\mathfrak{D}_k^{\sigma,\nu}$) we denote the union of all the components $D_{k,\lambda}^{\sigma,\nu}$ (respectively, $\mathfrak{D}_{k,\lambda}^{\sigma,\nu}$) of the set D bifurcating from the points $(\lambda, 0) \in I_k^{\sigma} \times \{0\}$ (respectively, $(\lambda, \infty) \in I_k^{\sigma} \times \{\infty\}$) with respect to the set $\mathbb{R} \times S_k^{\sigma,\nu}$.

Remark 1. It follows from [3, Theorem 3.1] that for each $k \in \mathbb{N}$, each σ and each ν the set $D_k^{\sigma,\nu}$ is contained in $\mathbb{R} \times S_k^{\sigma,\nu}$ and is unbounded in $\mathbb{R} \times E$.

Theorem 1. *For each $k \in \mathbb{N}$, each σ and each ν the set $\mathfrak{D}_k^{\sigma,\nu}$ is contained in $\mathbb{R} \times S_k^{\sigma,\nu}$ and either $\mathfrak{D}_k^{\sigma,\nu}$ meets $\mathbb{R} \times \{0\}$ for some $\lambda \in \mathbb{R}$, or the projection $P_{\mathbb{R}\times\{0\}}(\mathfrak{D}_k^{\sigma,\nu})$ of $\mathfrak{D}_k^{\sigma,\nu}$ onto $\mathbb{R} \times \{0\}$ is unbounded.*

Theorem 2. *If the set $\mathfrak{D}_k^{\sigma,\nu}$ meets $\mathbb{R} \times \{0\}$ for some $\lambda \in \mathbb{R}$, then $\lambda \in I_k^{\sigma}$, and if the set $D_k^{\sigma,\nu}$ meets $\mathbb{R} \times \{\infty\}$ for some $\lambda \in \mathbb{R}$, then $\lambda \in I_k^{\sigma}$.*

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On the approximate solution of one inverse problem

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The paper considers the inverse problem of determining an unknown function depending on a spatial variable on the right side of a parabolic equation. To approximately solve the problem posed, the method of successive approximations and the finite-difference method are successively adopted. The convergence of the approximate solution to the exact one is proved.

You need to find a pair $\{f(t), u(x,t)\}$ from the following relations:

$$u_t - u_{xx} = f(x)g(t), \quad (x,t) \in D = (0,1) \times (0,T], \tag{1}$$

$$u(x,0) = \varphi(x), \quad x \in [0,1], \tag{2}$$

$$(u,t) = \psi_0(t),\ u(1,t) = \psi_1(t),\ t \in [0,T], \tag{3}$$

$$\int_0^T u(x,t)dt = r(x), \quad x \in [0,1], \tag{4}$$

here $0 < T = const$, the given-functions $g(t), \varphi(x), \psi_0(t), \psi_1(t), r(x)$ satisfy the following conditions:

$A. g(t) \in C^{\alpha}[0,T],\ c_1\sqrt{T} \le \int_0^T g(t)dt \ \le c_2\sqrt{T}, 0 < c_1, c_2 = const,$
$\varphi(x) \in C^{2+\alpha}[0,1],\ \psi_0(t), \psi_1(t) \in C^{1+\alpha}[0,T], \varphi(0) = \psi_0(0),\ \varphi(1) = \psi_1(0),$
$r(x) \in C^{2+\alpha}[0,1]$.

Definition 1. The pair $\{f(t), u(x,t)\}$ is called a classical solution to problem (1)-(4), if :

1) $f(t) \in C^{\alpha}[0,T]$;
2) $u(x,t) \in C^{2+\alpha, 1+\alpha/2}(\overline{D})$,
3) for these functions conditions (1)-(4) are satisfied in the usual way .

It is proved that problem (1)-(4) is equivalent to the problem of determining $\{f(t), u(x,t)\}$ from conditions (1), (2), (3) and

$$f(x) = [u(x,T) - \varphi(x) - r_{xx}(x)] \,|\int\limits_0^T g(t)dt, \quad x \in [0,1], \tag{5}$$

For problem (1),(2)(3), (5) successive approximations are applied of following way: first select $f^{(0)}(x) \in C^{\alpha}[0,1]$ and solve the next problem for $s = 0$

$$u_t^{(s+1)} - u_{xx}^{(s+1)} = f^{(s)}(x)g(t), \quad (x,t) \in D, \tag{6}$$

$$u^{(s+1)}(x,0) = \varphi(x), \quad x \in [0,1] \tag{7}$$

$$u^{(s+1)}(0,t) = \psi_0(t), \quad u^{(s+1)}(1,t) = \psi_1(t), \quad t \in [0,T] \tag{8}$$

We find $u(x,t) \in C^{2+\alpha,1+\alpha/2}(\overline{D})$ [1] and further from the formula

$$f^{(s+1)}(x) = \left[u^{(s+1)}(x,T) - \varphi(x) - r_{xx}(x)\right] / \int\limits_0^T g(t)dt, \quad x \in [0,1], \tag{9}$$

We find the $f^(1)(x)$ and this function is used to carry out the next iteration.

Theorem 1. *Let: 1) functions $g(t), \varphi(x), \psi_0(t), \psi_1(t), r(x)$ satisfy condition A; 2) there is solution to problem (1), (2), (3), (5) by definition 1 .*

Then there is such a $0 < T^* \le T$ that in region $\overline{D}^* = [0,1] \times [0,T^*]$ the $\{f^{(s)}(t), u^{(s)}(x,t)\}$ functions found according to scheme (6)-(9) converge to the exact solution of problems (1), (2), (3), (5) at $s \to \infty$ with the speed of geometric progression.

In iteration (6)-(9) , at each step of finding a solution, $\{f^{(s)}(t), u^{(s)}(x,t)\}$ is carried out using a weighted finite -difference scheme [2]:

$$\frac{y_{j+1}^{(s+1)} - y_j^{(s+1)}}{\tau} + \Lambda\left[\sigma y_{j+1}^{(s+1)} + (1-\sigma)y_j^{(s+1)}\right] =$$

$$= f^{(s)}(x)\left[\sigma g_{j+1} + (1-\sigma)g_j\right], \quad (x_j, t_j) \in (\omega_h \times \omega_\tau), \tag{10}$$

$$y_0^{(s+1)} = \varphi(x_i), \quad x_i \in \overline{\omega}_h, \tag{11}$$

$$y_{j+1}^{(s+1)}(0) = \psi_0(t_{j+1}), \; y_{j+1}^{(s+1)}(1) = \psi_1(t_{j+1}), \quad t_j \in \overline{\omega}_\tau, \tag{12}$$

$$f^{(s+1)}(x_i) = \left[\, y_m^{(s+1)} - \varphi(x_i) + \Lambda r(x_i)\right] / \tau \sum_{j=0}^{m} g_j, \quad x_i \in \overline{\omega}_h, \tag{13}$$

here

$$\omega_h = \left\{x_i = ih, \quad h = 1/n, \quad i = \overline{0,n}\right\}, \quad \omega_\tau = \left\{t_j = i\tau, \ \tau = T/m, \quad j = \overline{0,m}\right\},$$

$$y_j = y(x_i, t_j) = u(x_i, t_j),$$

$$\Lambda y_j = -\frac{y_j(x_{i+1}) - 2y_j(x_i) + y_j(x_{i-1})}{h^2}.$$

First, the approximation error f the scheme is calculated and stability is shown based on the initial data and the right-hand side, and then the convergence of the scheme is proven.

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Uniqueness and assessment stability of the solution to the inverse problem for a semilinear equation of parabolic type

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By given functions $f(x,t,u)$, $\varphi(x)$, $\psi(x,t,u)$, $h(x)$ it is required to determine a pair of functions $\{c(x)\ , u(x,t)\}$ from the conditions:

$$u_t - \Delta u + c(x)\, u = f(x,t,u), \qquad (x,t) \in \Omega = D \times (0,T], \tag{1}$$

$$u(x,0) = \varphi(x)\,,\ x \in \bar{D} = D\bigcup \partial D, D \subset R^n, \tag{2}$$

$$\frac{\partial u}{\partial \overline{N}} = \psi(x,t,u)\,, \quad (x,t) \in S = \partial\, D \times [0,T]\,, \tag{3}$$

$$\int_0^T u(x,t)\,dt = h(x)\,,\ x \in \overline{D}, 0 < T = const. \tag{4}$$

We assume that the input data of the problem satisfies the following conditions:

$1^0.f(x,t,p) \in C^{\alpha,\alpha/2}\left(\bar{\Omega}\,\times\,R^1\right)$, there is a constant $m_1 > 0$, such that for all $p_1, p_2 \in R^1$ and $(x,t) \in \bar{\Omega}$ $|f(x,t,p_1) - f(x,t,p_2)| \le m_1 |p_1 - p_2|$;

2^0 $\varphi(x) \in C^{2+\alpha}\left(\bar{D}\right)$;

$3^0.\psi(x,t,p) \in C^{\alpha,\alpha/2}(S\,\times\,R^1)$, there is a constant $m_2 > 0$, such that for all $p_1, p_2 \in R^1$ and $(x,t) \in S$ $|\psi(x,t,p_1) - \psi(x,t,p_2)| \le m_2 |p_1 - p_2|$;

4^0. $h(x) \in C^{2+\alpha}\left(\bar{D}\right)$.

Definition 1. *A pair of functions* $\{c(x)\,,\, u(x,t)\}$ *is called a solution to problem* (1)-(4) *if*:

1) $c(x) \in C\left(\bar{D}\right)$

2) $u(x,t) \in C^{2,1}(\Omega)\bigcap C^{1,0}\left(\bar{\Omega}\right)$;

3) *these functions the relations of system* (1)-(4) *are satisfied, in this case, condition* (3) *is defined as follows*:

$$\frac{\partial u(x,t)}{\partial \overline{N}(x,t)} = \lim_{\substack{y \to x \\ y \in \sigma}} \frac{\partial u(y,t)}{\partial \overline{N}(x,t)},$$

where σ *is any closed cone with vertex that is contained in*$D\bigcup\{x\}$.

Definition 2. *The set*K^α *is called the well-posedness set for the problem* (1)-(4) if:

$$K^\alpha = \left\{(c,u)\,\middle|\,c(x) \in C^\alpha\left(\overline{D}\right)\,, \quad u(x,t) \in C^{2+\alpha,1+\alpha/2}\left(\bar{\Omega}\right)\,,\right.$$

there are constants $m_3,\ m_4\ > 0$ *such that*

$$\left.\left|D_x^l u(x,t)\right| \le m_3,\ l = 0,1,2,\ (x,t) \in \left(\bar{\Omega}\right), |c(x)| \le m_4,\ x \in\ \overline{D}\right\}.$$

Definition 3. *Let* $\{c_k(x),\ u_k(x,t)\}$ *be solutions of problem* (1)-(4) *corresponding to the data* $f_k(x,t,u),\ \varphi_k(x),\ \psi_k(x,t,u),\ h_k(x),\ k = 1,\ 2.$

We say that the solution of problem (1)-(4) *is stable if for any* $\varepsilon > 0$ *there is a*$\delta(\varepsilon) > 0$ *such that for*

$\|f_1 - f_2\|_0 < \delta$, $\|\varphi_1 - \varphi_2\|_2 < \delta$, $\|\psi_1 - \psi_2\|_0 < \delta$, $\|h_1 - h_2\|_1 < \delta$ *the inequality* $\|c_1 - c_2\|_0 + \|u_1 - u_2\|_0 < \varepsilon$.

The following theorem is proved:

Theorem. *Let*

1) $|h_k(x)| \geq const > 0$, $k = 1,\ 2$, $x \in \overline{D}$;

2) *functions* $f_k(x,t,u)$, $\varphi_k(x)$, $\psi_k(x,t,u)$, $h_k(x)$, $k = 1, 2$, *satisfy conditions* 1^0, 2^0, 3^0, 4^0 *respectively*;

3) *there exists solutions* $\{c_k(x),\ u_k(x,t)\}$, $k = 1, 2$, *to problems* (1)-(4) *and they belong to the set* K^α.

Then there exists a $T^* > 0$ *such that for* $(x,t) \in \bar{D} \times [0, T^*]$ *a solution for problem* (1)-(4) *is unique and the following estimate is true*:

$$\|c_1 - c_2\|_0 + \|u_1 - u_2\|_0 \leq M\left[\| f_1 - f_2\|_0 + \| \varphi_1 - \varphi_2\|_2 + \| \psi_1 - \psi_2\|_0 + \| h_1 - h_2\|_1\right],$$

where M>0 depends on the data of problem (1)-(4) *and the set* K^α.

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Global bifurcation of positive and negative solutions from infinity of certain nonlinear elliptic problems with indefinite weight

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Let Ω be a bounded domain in $\mathbb{R}^N$, $N > 1$, with a smooth boundary $\partial\Omega$. We consider the following nonlinear eigenvalue problem and let L be the differential operator

$$\begin{cases} Lu \equiv -\sum\limits_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_i}\right) + cu = \lambda a u + g(x,u,\nabla u,\,\lambda),\ x\in\Omega, \\ u=0,\ x\in\partial\Omega, \end{cases} \tag{1}$$

where λ is a real parameter, $\nabla u = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \ldots, \frac{\partial u}{\partial x_N}\right)$. We suppose that $a_{ij}(x) \in C^1\left(\overline{\Omega};\mathbb{R}\right)$, $i,j = 1,2,\ldots,N$, $c\,(x) \in C\left(\overline{\Omega};\mathbb{R}\right)$ and $c\,(x) \geq 0$, $x \in \overline{\Omega}$, $a(x) \in C\left(\overline{\Omega};\mathbb{R}\right)$ and takes both positive and negative values in Ω, and L is uniformly elliptic on $\overline{\Omega}$. The function $g \in C\left(\overline{\Omega}\times\mathbb{R}^{N+2};\mathbb{R}\right)$ and satisfies the following conditions:

$$u\,g(x,u,v,\lambda) \leq 0,\ (x,u,v,\lambda) \in \Omega\times\mathbb{R}^{N+2}\times\mathbb{R}; \tag{2}$$

there exists a positive constant K such that

$$|g(x,u,v,\lambda)| \leq M(|u|+|v|),\ (x,u,v,\lambda)\in\Omega\times\mathbb{R}^{N+2}\times\mathbb{R}; \tag{3}$$

for every bounded interval $\Lambda\subset\mathbb{R}$,

$$g(x,u,v,\lambda) = o(|u|+|v|) \ \text{ as } \ |u|+|v|\to+\infty, \tag{4}$$

uniformly in $x\in\Omega$ and in $\lambda\in\Lambda$.

Problems of type (1) arise in population genetics (see, g.e., [1]). In the case when $a_{ij}=1$ and $g(x,u,s,\lambda)=\lambda a(x)(1-u)$ the differential equation in

(1) is a reaction-diffusion equation, the real parameter λ corresponds to the reciprocal of the diffusion coefficient and the unknown function u represents a relative frequency [1].

The linear problem

$$\begin{cases} Lu(x) = \lambda a(x)u(x), \ x \in \Omega, \\ u(x) = 0, \ x \in \partial\Omega, \end{cases} \tag{5}$$

have positive and negative principal eigenvalues λ_1^+ and λ_1^-, respectively, which are simple. Moreover, the corresponding eigenfunction $u_1^\sigma(x)$, $x \in \bar{\Omega}$, $\sigma \in \{+, -\}$, can be chosen so that $u_1^\sigma(x) > 0$ for all $x \in \Omega$ and $\frac{\partial u_1^\sigma(x)}{\partial n} < 0$ for all $x \in \partial\Omega$ [2].

Let $E = \{u \in C^{1,\alpha}\left(\overline{\Omega}\right) : u = 0 \text{ on } \partial\Omega\}$ be the Banach space with the norm $|| \cdot ||_{1,\alpha} = || \cdot ||_{C^{1,\alpha}}$. A pair (λ, u) is called a solution to problem (1) if $u \in W^{2,p}(\Omega)$ and (λ, u) satisfies (1), where $\alpha \in (0,1)$ is given and p is a real number such that $p > N$ and $\alpha < 1 - N/p$. For this choice $W^{2,p}(\Omega)$ is compactly embedded in $C^{1,\alpha}(\overline{\Omega})$ (see [3]), and consequently, any solution of (1) belongs to $\mathbb{R} \times E$. Therefore, we will study the structure of the solutions set of (1) in $\mathbb{R} \times E$ (the norm in $\mathbb{R} \times E$ is defined as $||(\lambda, u)|| = \left\{|\lambda|^2 + ||u||_{1,\alpha}^2\right\}^{\frac{1}{2}}$).

By σ (ν) we denote an element of $\{+, -\}$ that is, either $\sigma = +$ (respectively, $\nu = +$) or $\sigma = -$ (respectively, $\nu = -$).

For each σ and each ν let $\mathcal{P}_\sigma^\nu$ be the set of functions $u \in E$ satisfying the following conditions:

(i) $\nu\, u(x) > 0$ for $x \in \Omega$;

(ii) $\nu \frac{\partial u(x)}{\partial n} < 0$ for $x \in \partial\Omega$;

(iii) $\sigma \int\limits_\Omega au^2 dx > 0$,

where $\frac{\partial u}{\partial n}$ is the outward normal derivative of u on $\partial\Omega$.

From the definition of the sets $\mathcal{P}_\sigma^+$ and $\mathcal{P}_\sigma^-$ it is seen that for each σ they are open disjoint subsets in E. Moreover, if $u \in \partial\mathcal{P}_\sigma^\nu$, then either there exists $x_0 \in \Omega$ such that $u(x_0) = 0$, or there exists $x_1 \in \partial\Omega$ such that $\frac{\partial u(x_1)}{\partial n} = 0$, or $\sigma \int\limits_\Omega au^2 dx = 0$ [2].

It follows from [2, Theorem 3.1] that for each σ and each ν there exists an unbounded component $\mathcal{C}_\sigma^\nu$ of the set of nontrivial solutions which meets $(\lambda_1^\sigma, \infty)$ and for which the following statements hold: (a) $\mathcal{C}_\sigma^\nu \subset \mathbb{R}^\sigma \times E$; (b) there exists a neighborhood $\mathcal{Q}_\sigma$ of $(\lambda_1^\sigma, \infty)$ such that $(\mathcal{C}_\sigma^\nu \cap Q_\sigma) \subset (\mathbb{R}^\sigma \times S_\sigma^\nu)$;

(c) either the set $\mathcal{C}_\sigma^\nu \setminus Q_\sigma$ meets $\mathbb{R}^\sigma \times \{\infty\}$ for some $\lambda \in \mathbb{R}$, or the set $\mathcal{C}_\sigma^\nu \setminus Q_\sigma$ meets $\mathcal{R}^\sigma = \mathbb{R}^\sigma \times \{0\}$ for some $\lambda \in \mathbb{R}$, or the projection of $\mathcal{C}_\sigma^\nu \setminus Q_\sigma$ onto $\mathcal{R}^\sigma$ is unbounded.

We introduce the following notations:

$$I_1^+ = [\lambda_1^+, \lambda_1^+ + M/a_0],\ I_1^- = [\lambda_1^- - M/a_0, \lambda_1^-],$$

where $a_0 = \min\limits_{x \in \bar{\Omega}} a(x)$.

Theorem 1. *Let conditions (2)-(4) and the following condition be satisfied:*

$$M < a_0 \min\{\lambda_2^+ - \lambda_1^+, \lambda_1^- - \lambda_2^-\}\,.$$

Then $\mathcal{C}_\sigma^\nu \setminus Q_\sigma$ meets $\mathbb{R}^\sigma \times \{\infty\}$ for some $\lambda \in I_1^\sigma \subset \mathbb{R}$. Furthermore, $\mathcal{C}_\sigma^\nu \subset I_1^\sigma \times E$.

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On the inverse boundary value problem for the beam equation with nonclassical boundary conditions

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In the domain

$$D_T = \{(x,t) : 0 \leq x \leq 1,\ 0 \leq t \leq T\},$$

consider the inverse boundary value problem for the equation

$$u_{tt}(x,t) - (q(x)u_x(x,t))_x + (p\,(x)u_{xx}(x,t))_{xx} = a(t)u(x,t) + f(x,t), \tag{1}$$

with initial conditions

$$u(x,0) = \varphi(x), u_t(x,0) = \psi(x),\ 0 \leq x \leq 1, \tag{2}$$

non-classical boundary conditions

$$u(0,t) = 0,\ \ 0 \leq t \leq T, \tag{3}$$

$$u_{xx}(0,t) = 0,\ \ 0 \leq t \leq T, \tag{4}$$

$$p\,(1)u_{xx}(1,t) + u_x(1,t) = 0,\ \ 0 \leq t \leq T, \tag{5}$$

$$\begin{gathered}(p\,(x)u_{xx}(x,t))_x\,|_{x=1} - q(1)u_x(1,t) - \\ ((q(x)\,u_x(x,t))_x - (p\,(x)u_{xx}(x,t))_{xx})\,|_{x=1} = 0,\ \ 0 < t < T,\end{gathered} \tag{6}$$

and an additional condition

$$u(x_0,t) = h(t),\ \ 0 \leq t \leq T, \tag{7}$$

where $x_0 \in (0,1)$ is a fixed point,

$$p\,(x) \in C^2([0,1]; (0,+\infty)),\ \ q(x) \in C^1([0,1]; [0,+\infty)),$$

$f(x,t)$, $\varphi(x)$, $\psi(x)$, $h(t)$ are are given functions, and $u(x,t)$, $a(t)$ are the sought functions.

Using the method of separation of variables, the spectral theory of fourth-order ordinary differential operators and analytical methods, the existence and uniqueness of a classical solution to problem (1)-(7) is proved (see [1, 2]).

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On one generalized spectral problem for perturbed bi-harmonic operator in cubic domain

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The work found eigenvalues and eigenfunctions for one perturbed bi-harmonic operator $(\partial_x^4 + \partial_y^4 + \partial_z^4)u \equiv [(-\Delta)^2 - 2\partial_x^2\partial_y^2 - 2\partial_y^2\partial_z^2 - 2\partial_z^2\partial_x^2]u = \lambda^2(-\Delta)u$ in cubic domain $\Omega = (0,l)^3 \subset \mathbb{R}^3$ with Dirichlet conditions $u = \partial_{\vec{n}}u = 0$ on the boundary of the cube $\partial\Omega$. A comparison of the found eigenvalues with the eigenvalues of the spectral problem for the bi-harmonic operator is established $(-\Delta)^2u = \lambda^2(-\Delta)u$ with Dirichlet conditions on the boundary. Note that the last problem arises when studying the bending oscillations of a clamped cubic body (building joint-ligament in the shape of a cube) and has applications in three-dimensional Stokes boundary problems describing the movement of fluid [1], structural mechanics, shipbuilding, etc.

We study the following generalized spectral problem

$$\left(\partial_x^4 + \partial_y^4 + \partial_z^4\right) u(x,y,z) = \lambda^2 \left(-\Delta u(x,y,z)\right), \quad (x,y,z) \in \Omega, \tag{1}$$

$$u(x,y,z) = \partial_{\vec{n}}u(x,y,z) = 0, \quad (x,y,z) \in \partial\Omega, \tag{2}$$

where $\Delta = \partial_x^2 + \partial_y^2 + \partial_z^2$, $\vec{n}$ is outer unit normal to $\partial\Omega$. Note that the spectral parameter λ^2 is located on the positive semi-axis of the real axis and is separated from zero.

Theorem 1. *The spectral problem for the perturbed bi-harmonic operator (1)–(2) has the following solution*

$$u_n(x,y) = X_n(x)Y_n(y)Z_n(z), \;\; (x,y,z)\in\Omega, \quad \lambda_n^2, \quad n\in\mathbb{N}, \tag{3}$$

where up to a constant factor (and ordered according to the numbers λ_n^2):

$$\begin{cases} X_{2n-1}(x) = \sin^2\frac{\lambda_{2n-1}x}{2}, \;\; \lambda_{2n-1}^2 = \left(\frac{(2n-1)\pi}{l}\right)^2, \\ X_{2n}(x) = A_{2n}\sin^2\frac{\lambda_{2n}x}{2} - B_{2n}\left[\lambda_{2n}x - \sin\lambda_{2n}x\right], \;\; \lambda_{2n}^2 = \left(\frac{2\nu_n}{l}\right)^2, \end{cases} \quad n\in\mathbb{N}, \tag{4}$$

$$\begin{cases} Y_{2n-1}(y) = \sin^2\frac{\lambda_{2n-1}y}{2}, \;\; \lambda_{2n-1}^2 = \left(\frac{(2n-1)\pi}{l}\right)^2, \\ Y_{2n}(y) = A_{2n}\sin^2\frac{\lambda_{2n}y}{2} - B_{2n}\left[\lambda_{2n}y - \sin\lambda_{2n}y\right], \;\; \lambda_{2n}^2 = \left(\frac{2\nu_n}{l}\right)^2, \end{cases} \quad n\in\mathbb{N}, \tag{5}$$

$$\begin{cases} Z_{2n-1}(z) = \sin^2\frac{\lambda_{2n-1}z}{2}, \;\; \lambda_{2n-1}^2 = \left(\frac{(2n-1)\pi}{l}\right)^2, \\ Z_{2n}(z) = A_{2n}\sin^2\frac{\lambda_{2n}z}{2} - B_{2n}\left[\lambda_{2n}z - \sin\lambda_{2n}z\right], \;\; \lambda_{2n}^2 = \left(\frac{2\nu_n}{l}\right)^2, \end{cases} \quad n\in\mathbb{N}, \tag{6}$$

$\{\lambda_n^2, \;\; n\in\mathbb{N}\}$ *is an ordered sequence of nondecreasing eigenvalues is formed by the roots of the equation:*

$$\lambda\left\{4\sin^4\frac{\lambda}{2} - [\lambda - \sin\lambda]\sin\lambda\right\} = 0, \tag{7}$$

which is equivalent to the following relations:

$$\lambda_n \neq 0, \quad \begin{cases} \sin\frac{\lambda_{2n-1}l}{2} = 0, \quad \lambda_{2n-1}^2 = \left(\frac{(2n-1)\pi}{l}\right)^2, \\ \tan\frac{\lambda_{2n}l}{2} = \frac{\lambda_{2n}l}{2}, \;\; \lambda_{2n}^2 = \left(\frac{2\nu_n}{l}\right)^2, \end{cases} \quad n\in\mathbb{N}, \tag{8}$$

where $A_n = \lambda_{2n}l - \sin\lambda_{2n}l$, $B_n = \sin^2\frac{\lambda_{2n}l}{2}$, and $\{\nu_n, \;\; n\in\mathbb{N}\}$ be positive roots of the equation $\tan\nu = \nu$.

In applications, firstly, when studying the vibrations of a clamped curved plate with commensurate thickness, it becomes necessary to solve the following spectral problem for the unperturbed bi-harmonic operator

$$(-\Delta)^2 u(x,y,z) = \lambda^2(-\Delta)u(x,y,z), \;\; (x,y,z)\in\Omega, \tag{9}$$

$$u(x,y,z) = \partial_{\vec{n}} u(x,y,z) = 0, \ \ (x,y,z) \in \partial\Omega. \tag{10}$$

Secondly, when passing from the stationary three-dimensional Stokes problem with respect to the 3-D vector of fluid velocity and pressure (with Dirichlet conditions on the boundary) to the equation with respect to the scalar stream function, we obtain the Dirichlet boundary value problem for the bi-harmonic equation [1-2]. Here there is also a need to study the spectral problem (9)–(10).

Theorem 2. *Let $\lambda_n^2,\ n \in \mathbb{N}$ be the eigenvalues (3)–(8) of spectral problems for the perturbed bi-harmonic operator (1)–(2), and $\bar{\lambda}_n^2,\ n \in \mathbb{N}$ be the eigenvalues of spectral problems for the unperturbed bi-harmonic operator (9)–(10). These eigenvalues are related by the inequalities: $\lambda_n^2 \le \bar{\lambda}_n^2,\ \ n \in \mathbb{N}$.*

For spectral problems (9)–(10) with operator A, (1)–(2) with operator $A^{[1]}$ and the spectral problem with the operator $A^{[2]}$:

$$2\left(\partial_x^2\partial_y^2 + \partial_y^2\partial_z^2 + \partial_z^2\partial_x^2\right) u(x,y,z) = \mu^{[2]}(-\Delta)u(x,y,z), \ \ (x,y,z) \in \Omega, \tag{11}$$

$$u(x,y,z) = 0, \ \ (x,y,z) \in \partial\Omega, \tag{12}$$

the following theorem holds.

Theorem 3. *Eigenvalues $\mu_{n+\max\{j,k,m\}-1}$ of the unperturbed bi-harmonic operator A, eigenvalues λ_n^2 of the perturbed bi-harmonic operator $A^{[1]}$, and eigenvalues $\mu^{[2]}$ of the operator $A^{[2]}$ (11)–(12) satisfy the inequalities*

$$\lambda_n^2 = \left\{ \begin{array}{c} \left((2n-1)\pi/l\right)^2 \\ \\ \left(2\nu_n/l\right)^2 \end{array} \right. + \frac{2\left(j^2k^2 + k^2m^2 + j^2m^2\right)\pi^2}{\left(j^2+k^2+m^2\right)l^2} \le \mu_{n+\max\{j,k,m\}-1}, \tag{13}$$

$n, j, k, m \in \mathbb{N}$, where ν_n be positive roots of the equation $\tan\nu = \nu$, and the numbers λ_n^2 be ordered in non-decreasing order.

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On approximate solutions of the Cauchy problem for elliptic systems of the first order

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In this paper we are talking about an approximate solution of the Cauchy problem for matrix factorizations of the Helmholtz equation in a multidimensional bounded domain.

The theory of ill-posed problems is a direction of mathematics which has developed intensively in the last two decades and is connected with the most varied applied problems: interpretation of readings of many physical instruments and of geophysical, geological, and astronomical observations, optimization of control, management and planning, synthesis of automatic systems, etc. The concept of a well-posed problem is connected with investigations by the famous French mathematician Hadamard of various boundary value problems for the equations of mathematical physics [5].

In many well-posed problems for systems of equations of elliptic type of the first order with constant coefficients that factorize the Helmholtz operator, it is not possible to calculate the values of the vector function on the entire boundary. Therefore, the problem of reconstructing the solution of systems of equations of first order elliptic type with constant coefficients, factorizing the Helmholtz operator (see, for instance [1]-[4]), is one of the topical problems in the theory of differential equations.

Let $\mathbb{R}^m$ be the $m-$dimensional real Euclidean space,

$$\zeta = (\zeta_1, \ldots, \zeta_m) \in \mathbb{R}^m, \quad \eta = (\eta_1, \ldots, \eta_m) \in \mathbb{R}^m,$$

$$\zeta' = (\zeta_1, \ldots, \zeta_{m-1}) \in \mathbb{R}^{m-1}, \quad \eta' = (\eta_1, \ldots, \eta_{m-1}) \in \mathbb{R}^{m-1}.$$

Next, we use the following notation:

$$r = |\eta - \zeta|\,, \quad \alpha = |\eta' - \zeta'|\,, \quad z = i\sqrt{a^2+\alpha^2} + \eta_m, \quad a \geq 0,$$

$$\partial_\zeta = (\partial_{\zeta_1}, \dots, \partial_{\zeta_m})^T\,, \quad \partial_\zeta = \chi^T, \quad \chi^T = \begin{pmatrix} \chi_1 \\ \dots \\ \chi_m \end{pmatrix} - \text{transposed vector} \quad \chi,$$

$$W(\zeta) = (W_1(\zeta), \dots, W_n(\zeta))^T, \quad v^0 = (1, \dots, 1) \in \mathbb{R}^n, \quad n = 2^m, \quad m \geq 2,$$

$$E(w) = \begin{Vmatrix} w_1 & 0 & \cdots & 0 \\ 0 & w_2 & \cdots & 0 \\ \dots & \dots & \ddots & \dots \\ 0 & 0 & 0 & w_n \end{Vmatrix} - \text{diagonal matrix}, w = (w_1, \dots, w_n) \in \mathbb{R}^n.$$

We also consider a bounded simply-connected domain $\Omega \subset \mathbb{R}^m$, having a piecewise smooth boundary $\partial\Omega = \Sigma \bigcup D$, where Σ is a smooth surface lying in the half-space $\eta_m > 0$ and D is the plane $\eta_m = 0$.

Let us consider the following first order systems of linear partial differential equations with constant coefficients

$$P\left(\partial_\zeta\right) W(\zeta) = 0, \tag{1}$$

in the domain Ω, where $P\left(\partial_\zeta\right)$ is the matrix differential operator of the first-order.

Also consider the set

$$S\left(\Omega\right) = \left\{W : \overline{\Omega} \longrightarrow \mathbb{R}^n\right\},$$

here W is continuous on $\overline{\Omega} = \Omega \cup \partial\Omega$ and W satisfies the system (1).

Formulation of the problem. *Suppose $W(\eta) \in S(\Omega)$ and*

$$W(\eta)|_\Sigma = f(\eta), \quad \eta \in \Sigma.$$

Here, $f(\eta)$ a given continuous vector-function on Σ. It is required to restore the vector function $W(\eta)$ in the domain Ω, based on it's values $f(\eta)$ on Σ.

Theorem *Let $W(\eta) \in S(\Omega)$ it satisfy the inequality*

$$|W(\eta)| \leq M, \quad \eta \in D.$$

If

$$W_\sigma(\zeta) = \int\limits_{\Sigma} L_\sigma(\eta, \zeta; \lambda) W(\eta) ds_y, \quad \zeta \in \Omega,$$

then the following estimates are true

$$|W(\zeta) - W_\sigma(\zeta)| \leq MK(\lambda, \zeta)\sigma^k e^{-\sigma\zeta_m}, \quad \sigma > 1, \quad m = 2k,\, k \geq 1, \quad \zeta \in \Omega,$$

$$|W(\zeta) - W_\sigma(\zeta)| \leq MK(\lambda, \zeta)\sigma^{k+1} e^{-\sigma\zeta_m}, \quad \sigma > 1, \quad m = 2k{+}1,\, k \geq 1, \quad \zeta \in \Omega.$$

Here functions bounded on compact subsets of the domain Ω, we denote by $K(\lambda, \zeta)$.

Corollary For each $\zeta \in \Omega$, the following equality holds

$$\lim_{\sigma \to \infty} W_\sigma(\zeta) = W(\zeta).$$

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Stabilization of solutions to nonlinear parabolic and damped wave equations

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The talk will be devoted to the problem of global existence and blow up in a finite time of solutions with positive initial energy of initial boundary value problems for nonlinear parabolic amd wave equations

$$\begin{cases} u_{tt} + bu_t + Lu = f_1(u,v), \\ v_t + Mv = f_1(u,v), \end{cases}$$

where L and M are second order elliptic operator, the functions $f_1(\cdot,\cdot), f_2(\cdot,\cdot)$ are so that $F(u,v) := [f_1(u,v), f_2(u,v)], u, v \in \mathbb{R}$ is a continuous and conservative vector field in $\mathbb{R}^2$, i.e. there exists a function $G(u,v) : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\nabla G(u,v) = [f_1(u,v), f_2(u,v)]$.

Solution of a boundary value problem for the heat equation with discontinuous coefficient by Fourier method

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In this paper, the Fourier method is used to justify the solution of an initial boundary value problem for the heat equation with a discontinuous coefficient and a general conjugation condition. Using the method of separation of variables, this problem is reduced to the corresponding non-self-adjoint spectral problem. The eigenvalues and eigenfunctions of this spectral problem are found and it is proved that the system of eigenfunctions forms a Riesz basis. Next, a theorem on the existence and uniqueness of a classical solution to the problem posed is proven.

We consider the initial boundary value problem for the heat conduction equation with a discontinuous coefficient:

$$\frac{\partial u}{\partial t} = a_i^2 \frac{\partial^2 u}{\partial x^2}, \tag{1}$$

in domain $\Omega = \cup \Omega_i, \quad \Omega_1 = \{(x,t) : l_0 < x < l_1,\ 0 < t < T\}$,
$\Omega_2 = \{(x,t) : l_1 < x < l_2, 0 < t < T\}$,$(i = 1,2)$ with initial conditions

$$u(x,0) = \varphi(x), \quad l_0 \le x \le l_2, \tag{2}$$

boundary conditions of the form

$$u(l_0,t) = u(l_2,t) = 0, \quad 0 \le t \le T, \tag{3}$$

and with general interface conditions

$$\begin{cases} k_1 \frac{\partial u(l_1-0,t)}{\partial x} = h\left(\theta u(l_1+0,t) - u(l_1-0,t)\right), \\ k_1 \frac{\partial u(l_1-0,t)}{\partial x} = k_2 \frac{\partial u(l_1+0,t)}{\partial x}, \end{cases} \tag{4}$$

where coefficients $a_i^2 = \frac{k_i}{c_i \rho_i}, \quad (i = 1,2), \quad h > 0, \quad \theta > 0$, k_i .- thermal conductivity coefficient, c_i - specific heat capacity, ρ_i - density, a_i^2 - thermal diffusivity coefficient.

The following is proven

Theorem. *Let $\varphi(x)$ - a twice continuously differentiable function satisfying the conditions* $\varphi(l_0) = \varphi(l_2) = 0$, $k_1\varphi'(l_1-0) = h\left(\theta\varphi(l_1+0) - \varphi(l_1-0)\right)$, $k_1\varphi'(l_1-0) = k_2\varphi'(l_1+0)$. *Then the function* $u(x,t) = \sum\limits_{n=1}^{\infty} \varphi_n X_n(x) e^{-\lambda_n t}$, *is the only classical solution to problem (1)-(4), where the coefficients* $\varphi_n -$*are determined by the formula* $\varphi_n = \int\limits_{l_0}^{l_2} \varphi(x) Y_n(x) dx$, $X_n(x) -$ *eigenfunctions of the corresponding spectral problem,* $Y_n(x) -$ *eigenfunctions of the conjugate problem.*

On the existence of the fundamental transfer matrix in two-dimensional potential scattering and the propagating-wave approximation

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Potential scattering admits a formulation in terms of a fundamental notion of transfer matrix which is a linear operator admitting a Dyson series expansion involving an effective non-self-adjoint Hamiltonian operator. This approach to potential scattering has led to a number of interesting developments in two and three dimensions such as construction of the first examples of short-range potentials for which the first Born approximation is exact as well as potentials that display broadband invisibility. This talk presents a first step towards a rigorous proof of the existence of the fundamental transfer matrix in two dimensions. It offers a solution of this problem within the context of propagating-wave approximation. This is an approximation scheme that ignores the contribution of the evanescent waves to the scattering data and is valid for high energies and weak potentials.

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Optimal control of hyperbolic type differential inclusions

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The paper discusses the optimization of 2-D wave differential inclusions (DFIs) with the Laplacian in a bounded parallelepiped and the first mixed initial-boundary value problem. Particular attention is paid to problems with state constraints, for which optimality conditions are formulated in terms of the Euler-Lagrange adjoint inclusions. The problem of optimal control of ordinary [1]-[6] and partial differential equations/inclusions [2, 3, 6, 7] occurs in many applications, from engineering to science in economic dynamics, classical optimal control problems, differential games, etc. Hyperbolic differential equations/inclusions arise in many applied problems such as string vibrations, acoustic modelling, supersonic fluid flows, etc. The necessary definitions and concepts are given from the book [3]; let $\langle v,u \rangle$ is a scalar product of a pair $(v,u) \in \mathbb{R}^n$. In terms of the Hamiltonian function, LAM in the "non-convex" case is defined as $F^*(u^*;(\tilde{v},\tilde{u})) := \{v^* : H_F(v,u^*) - H_F(\tilde{v},u^*) \leq \langle v^*, v-\tilde{v} \rangle, \forall v \in \mathbb{R}^n\}$. Note that, the concept of LAM given in the article is closely related to the concept of co-derivative concept of [7], which is essential in the non-convex case. We consider the optimal control of 2-D wave DFIs with state constraints:

$$\text{minimize } J[v(\cdot,\cdot,)] = \int_0^T \iint_Q f(v(x,y,t),x,y,t)\,dxdydt,$$

$$(WHP) \qquad \begin{aligned} & v_{tt}(x,y,t) - \nabla^2 v(x,y,t) \in F(v(x,y,t),x,y,t),\ (x,y,t) \in Q \times [0,T], \\ & v(x,y,t) \in \Omega(x,y,t),\ (x,y,t) \in Q \times [0,T],\ Q = [0,L] \times [0,S], \\ & v(x,y,0) = \alpha_1(x,y),\ v_t(x,y,0) = \alpha_2(x,y),\ v(x,0,t) = \beta_0(x,t), \\ & v(x,S,t) = \beta_S(x,t),\ v(0,y,t) = \gamma_0(y,t),\ v(L,y,t) = \gamma_L(y,t), \end{aligned}$$

where $F(\cdot,x,y,t)$ and Ω are a convex set-valued mappings, $f(\cdot,x,y,t)$ is proper convex function, ∇^2 is a 2-D Laplacian: $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$ and α_1, α_2 and

$\beta_0, \beta_S, \gamma_0, \gamma_L$ are given continuous functions, L, S, T are real positive numbers. We assume throughout the context that feasible solutions are classical solutions. To formulate the theorem on the optimality condition, we first introduce the hyperbolic type DFI with self-adjoint Laplacian, the associated argmaximum conditions and a homogeneous "endpoint" and boundary conditions:

$$(i)\ v^*_{tt}(x,y,t) - \nabla^2 v^*(x,y,t) \in F^*(v^*(x,y,t); (\tilde{v}(x,y,t), \tilde{v}_{tt}(x,y,t)$$
$$-\nabla^2 \tilde{v}(x,y,t)), x, y, t) - \partial f(\tilde{v}(x,y,t), x, y, t) + K^*_\Omega(\tilde{v}(x,y,t).$$
$$(ii)\ v^*(x,y,T) = 0,\ v^*_t(x,y,T) = 0,\ v^*(x,0,t) = v^*(x,S,t)$$
$$= v^*(0,y,t) = v^*(L,y,t) = 0, (x,y,t) \in Q \times [0,T].$$
$$(iii)\ \tilde{v}_{tt}(x,y,t) - \nabla^2 \tilde{v}(x,y,t) \in F_A\left(\tilde{v}(x,y,t); v^*(x,y,t), x, y, t\right).$$

Theorem 1 *For $\tilde{v}(x,y,t) \in \Omega(x,y,t)$ to be optimal in problem (WHP) with initial and boundary conditions and 2-D Laplacian hyperbolic DFI, it is sufficient to have a classical solution $v^*(x,y,t)$ satisfying conditions (i)-(iii) with adjoint DFI and homogeneous endpoint and boundary conditions.*
The formulated problem (WHP) can be generalized to the case with second order hyperbolic multidimensional DFIs.

Let us write the matrix form of the problem in the form of a hyperbolic polyhedral DFI:

$$(PHP) \quad Av(x,y,t) - B\left(v_{tt}(x,y,t) - \nabla^2 v(x,y,t)\right) \le d,$$
$$v(x,y,t) \in \Omega,\ \Omega = \{v : Dv \le c\},\ (x,y,t) \in Q \times [0,T],$$
$$v(x,y,0) = \alpha_1(x,y),\ v_t(x,y,0) = \alpha_2(x,y),\ v(x,0,t) = \beta_0(x,t),$$
$$v(x,S,t) = \beta_S(x,t), v(0,y,t) = \gamma_0(y,t), v(L,y,t) = \gamma_L(y,t), Q = [0,L] \times [0,S],$$

where A, B, D are $s \times n$ dimensional matrices, d, c are s-dimensional column-vector, $f(\cdot, x, y, t)$ is a polyhedral function, i.e. epi f is a polyhedral set in $\mathbb{R}^{n+1}$. On the basis of Theorem 1 we have:

$$B^*(q^*_{tt}(x,y,t) - \nabla^2 q^*(x,y,t)) - A^* q(x,y,t) \in \partial f(\tilde{v}(x,y,t), x, y, t) - D^*\gamma(x,y,t),$$
$$\left\langle A\left(\tilde{v}_{tt}(x,y,t) - \nabla^2 \tilde{v}(x,y,t)\right) - B\tilde{v}(x,y,t) - d, q(x,y,t)\right\rangle = 0, \tag{1}$$
$$B^* q^*(x,y,T) = 0,\ B^* q^*_t(x,y,T) = 0,\ q(x,y,t) \ge 0,\ \gamma(x,y,t) \ge 0$$
$$B^* q^*(x,0,t) = B^* q^*(x,S,t) = B^* q^*(0,y,t) = B^* q^*(L,y,t) = 0. \tag{2}$$

Theorem 2 *Suppose that $f(\cdot, x, y, t)$ is a polyhedral function and that F is a polyhedral mapping. Then, in order for the solution $\tilde{v}(x,y,t)$, to be optimal*

in the problem (PHP), it is sufficient that there exist functions $q(x,y,t) \geq 0, \gamma(x,y,t) \geq 0$ *satisfying (1) and (2).*
Example 1 Consider a problem with 2-D wave DFI and with $f(v,x,y,t) \equiv v$:

$$v_{tt}(x,y,t) - \nabla^2 v(x,y,t) \geq 0, Q = [0,\pi] \times [0,\pi],$$
$$v(x,y,0) = \sin x \cdot \sin y, v_t(x,y,0) = 0,$$
$$v(x,0,t) = 0, v(0,y,t) = 0, v(\pi,y,t) = 0, v(x,\pi,t) = 0.$$

which admits the exact fsolution $\tilde{v}(x,y,t) = \sin x \cdot \sin y \cdot \cos \sqrt{2}t$ and $J[v(\cdot,\cdot,\cdot)] = \left(4/\sqrt{2}\right) sin\sqrt{2}\pi$.

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On blow-up of solution for one nonlinear problem

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We consider the following problem

$$u_n - \sum_{i=1}^{n} D_i(|D_i u|^{p-2} D_i u) - \alpha \Delta u_i + f(u) = 0, \quad (x,t) \in \Omega \times [0,T], \tag{1}$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega, \tag{2}$$

$$\sum_{i=1}^{n} D_i(|D_i u|^{p-2} D_i u) \cos(x_i, v) + \alpha \frac{\partial u_t}{\partial n} = g(u), \quad (x,t) \in \partial\Omega \times [0,T], \tag{3}$$

where $\Omega \subset \mathbb{R}^n$, $n \geq 2$ is a boundary domain with smooth boundary $\partial\Omega$, $u_0(x) \in W_2^1(\Omega)$, $u_1(x) \in L_2(\Omega)$ are given functions, $f(u)$ and $g(u)$ are some nonlinear functions, α is positive number, $p \geq 2$. $D_i = \frac{\partial}{\partial x_i}$, $i = 1, 2, ..., n$, $\frac{\partial}{\partial n}$ - the external normal in $\partial\Omega$.

A lot of works (for example, see [1]- [5]) were devoted to the problems of solutions behavior for particular cases of question of type (1) with different conditions. In general, these works deal with the nonlinearity presented in the equation.

In this work, we study a blow-up of solution for a problem (1)-(3), when boundary function has some smoothing properties.

Theorem 1. *Let's for any $u \in \mathbb{R}$ and for some α satisfies following conditions*

$$2(2\alpha+1)F(u) - uf(u) \geq 0, \quad F(u) = \int_0^u f(s)ds,$$

$$ug(u) - 2(2\alpha+1)G(u) \geq 0,$$

$$\int_\Omega F(u_0)dx - \int_{\partial\Omega} G(u_0)ds + \frac{1}{p}\int_\Omega \sum_{i=1}^{n} |D_i u_0|^p dx \leq 0, \quad (u_0, u_1) > 0.$$

Then, if the problem (1)-(3) has a solution $u(x,t) \in W_2^1(0,T;W_2^2(\Omega)) \cap W_2^2(0,T;L_2(\Omega))$, *then exists* $t_0 < \infty$ *such that*

$$\lim_{t\to t_0}\left[\|u(x,t)\|^2 + \int_0^t \|\nabla u(x,\tau)\|^2 d\tau\right] = \infty$$

holds.

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Cauchy problem for a class of hyperbolic equation with integral nonlinearity and dissipation

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The existence of a global solution of the Cauchy problem for a class of integral nonlinear quasilinear hyperbolic equations with an anisotropic elliptic part is proved. A plot of the decay rates of the corresponding solutions is determined. $(0,\infty)\times R^n$ for a quasi-linear hyperbolic equation with an integral nonlinear anisotropic elliptic part in the domain:

$u_{tt}+u_t+\sum_{i=1}^{n}(-1)^{l_i}a_i(t,x,[u]_{l,i})D^{2l_i}_{x_i}=0$ (1)

$u(x,0)=\varphi(x),u_t(0,x)=\psi(x)$ (2)

Let's look at the Cauchy problem. Here $l_1,l_2,...,l_n\in\{1,2,3,...\}$, $[u]_{l,i}=\sum_{k=1}^{n}\beta_{ik}\int_{R^n}\left|D^{l^k}u\right|^2dx,\beta_{ik}\in R;i,k=1,2,3,...$ In a special case $l_1=l_2=...=l_n=1$ and $[u]_{l,i}=\int_\Omega|\nabla u|^2dx$ when (1) is the Kirchhoff equation with known dissipation.

Assuming the following conditions are satisfied:

1) $a_i(t,\xi)=1+a_{1i}(t,\xi),i=1,2,...,n$ so that $a_i(t,\xi)\,[0,\infty)\times(b,b),l>0$ in the area t and ξ are functions differentiated with respect to their variables.$a_{1_{i\xi}}(t,\xi)$ t is differentiable with respect to the variable.

2) All $t\in[0,\infty),\xi\in(-b,b)$ the following inequalities hold for: $|a_{1_i}(t,\xi)|\le c|\xi|^p;\left|a_{1_{i_t}}(t,\xi)\right|\le c|\xi|^p;\left|a_{1_{i\xi}}(t,\xi)\right|\le c|\xi|^{p-1}$ Here $p>1$ and $c>0$ are certain constants.

$H_2=W_2^{(r+1)l}(R_n)\times W_2^{(rl)}(R_n)$ with $\langle w^1,w^2\rangle_{H_r}=\langle u^1,u^2\rangle_{W_2^{(r+1)l}(R_n)}+\langle v_1,v_2\rangle_{W^{rl}(R_n)}$ let's denote the space with the scalar product. Here $w^1=\begin{pmatrix}u^1\\v^1\end{pmatrix}$,$w^2=\begin{pmatrix}u^2\\v^2\end{pmatrix}$, and $r=0,1$. Appropriate norm $\|\cdot\|_{H_r}=\sqrt{\langle\cdot,\cdot\rangle}_{H_r}$ denote by U_δ^r with H_r Let us denote the sphere centered at zero in

$U^r_{\delta_1}=\left\{\left(v_1^1,v_2^2\in H_r:\|v_1^1\|^2_{W_2^{(r+1)l}(R_n)}+\|v_2^2\|^2_{W_2^{rl}(R_n)}<\delta^2\right)\right\},r=0,1$

Theorem 1. *Terms of detention 1), 2) are satisfied and so be it $\delta>0$ there is any $(\varphi,\psi)\in U^r_{\delta_0}$ for (1), (2) is the only one*

$u \in C\left([0,\infty);W_2^{2l}(R_n)\right)\bigcap C^1\left([0,\infty);W_2^{l}(R_n)\right)\bigcap C^2\left([0,\infty);L_2(R_n)\right)$ *has a solution and* $u(t,x)$ *The following statements are true for:*

$\|D^\alpha u(t,\cdot)\|_{L_2(R_n)} \le C_{\delta_0}(1+t)^{-\left|\frac{\alpha}{l}\right|}$ *here* $\alpha=(\alpha_1,...,\alpha_n)\,;\alpha_1,...,\alpha_n\in N\cap\{0\}$ $\left|\frac{\alpha}{l}\right|=\frac{\alpha_1}{l_1}+\frac{\alpha_2}{l_2}+...+\frac{\alpha_n}{l_n}$, $c_{\delta_0}<0$

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Dirichlet problem for a non-uniformly elliptic equation with small terms

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This topic on the existence of solutions of uniformly elliptic equations was proved by Stampacia [6]. On study the L^1 or measure data Dirichlet problem for elliptic and parabolic (also nonlinear) equations we quote the series of works by Boccardo and his coauthors [1]. On the Sobolev-Poincare type inequalities of non-uniform gradient see [3, 4, 5]. In this note, we have considered the analogous problem for the non-uniform elliptic equations.

On this note, for the elliptic equations

$$\begin{cases} -\frac{\partial}{\partial z_i}\left(a_{ij}\left(t,z\right)\frac{\partial u}{\partial z_j}\right)+a_i(z)\frac{\partial u}{\partial x_i}+b_j(z)\frac{\partial u}{\partial y_j}=f\left(z\right),\,(t,z)\in Q_T, \\ u\Big|_{\partial\Omega}=0, \end{cases} \tag{1}$$

satisfying the non-uniformly ellipticity condition

$$c_1(\omega(x)\,|\xi|^2 + |\eta|^2) \le A(z)\zeta\cdot\zeta \le c_2(\omega(x)\,|\xi|^2 + |\eta|^2) \tag{2}$$

the very weak solvability is established for the data $f \in L^1(D), a_i/\sqrt{\omega}, b_j \in L^{N+\epsilon}(D)$. Where $\Omega \subset \mathbb{R}^N$ is a bounded domain with boundary $\partial\Omega$. The equation we considered is elliptic since matrix $A(z) = \|a_{ij}(t,z)\|$ is positively definite in $D = \{z = (x,y) \in \Omega : \, x \in \mathbb{R}^n, \, y \in \mathbb{R}^m\}$ and $\forall \zeta = (\xi,\eta) \in \mathbb{R}^N$, $\xi \in \mathbb{R}^n$, $\eta \in \mathbb{R}^m$, $N = n+m$; $m,n \ge 1$. The weight function $\omega(x)$ in (2) belongs to some $A_2(\mathbb{R}^n)$ -class by Muckenhoupt.

$$\left(\int_K \omega\,dx\right)\left(\int_K \omega^{-1}\,dx\right) \le \alpha|Q|^2, \tag{3}$$

for all Euclidean cubes $K \subset \mathbb{R}^n$ with edges parallel to the coordinate exes, $|K|$ denotes n-dimensional Lebesgue measure of the cube K. The constant $\alpha > 0$ does not depend on K.

Definition. A solution of the problem (1) is defined using the distributional approach. We say, the function $u : \bar{D} \to \mathbb{R}$ is a weak solution of problem (1) if $\omega(x)\frac{\partial u}{\partial x_i}, \frac{\partial u}{\partial y_j} \in L^1(D))$ and such that $\forall \varphi \in Lip(\bar{D}), \, \varphi(z)\Big|_{\partial D} = 0$ it holds

$$\int\limits_{Q_T} a_{ij}(z)\frac{\partial u}{\partial z_i}\frac{\partial \varphi}{\partial z_j}\,dz + \int_D a_i(z)\frac{\partial u}{\partial x_i}\varphi\,dz + \int_D b_j(z)\frac{\partial u}{\partial y_j}\varphi\,dz$$

$$= \iint\limits_{Q_T} f(z)\,\varphi(z)\,dz. \tag{4}$$

Theorem 1. *Let (2) be satisfied, where $\omega \in A_2(\mathbb{R}^n)$-class function satisfying also the next condition*

$$\Big(\int_{K_r^x} \omega(s)ds \Big/ \int_{K_R^x} \omega(s)ds\Big)^{\frac{1}{2}-\frac{m}{2}\left(\frac{1}{2}-\frac{1}{q}\right)} \ge C\,(r/R)^{1-\frac{m(n+2)}{2}\left(\frac{1}{2}-\frac{1}{q}\right)} \tag{5}$$

for all $r \in (0,R)$, $x \in K_R^a$ is satisfied by some $q \ge 2$. Let $f \in L^1(D)$ and $a_i(z)/\sqrt{\omega(x)}$, $i = \overline{1,n}$, $b_j(z) \in L^{2q/(q-2)+\epsilon}(D)$, $j = \overline{1,m}$, $c(z) \in L^{q/(q-2)+\epsilon}(D)$. Then for the condition

$$\|a(z)/\sqrt{\omega}\|_{L^{2q/(q-2)+\epsilon}} + \|b(z)\|_{L^{2q/(q-2)+\epsilon}} +, \ \|c(z)\|_{L^{q/(q-2)+\epsilon}} < \delta$$

with $\delta > 0$ *being sufficiently small, there exists a very weak solution* $u(z)$ *of the problem (1) in the sense (4) such that*

$$u \in L_{q/2-\epsilon}(D) \quad and \quad \omega(x)^{1/2}|\nabla_x u|,\, |\nabla_y u| \in L_{2q/(q+2)-\epsilon}(D)$$

for sufficiently small $\epsilon > 0$. *Moreover, the estimate fulfilled*

$$\|u\|_{L_{q/2-\epsilon}(D)} + \|\omega^{1/2}|\nabla_x u| + |\nabla_y u|\|_{L_{2q/(q-2)-\epsilon}(D)} \leq C\|f\|_{L_1(D)},$$

the constant C *depends on* $n, m, a_i, b_j, \omega, C_1, C_2$.

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Global bifurcation from infinity in nonlinearizable eigenvalue problems for some fourth-order ordinary differential equations

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We consider the following nonlinear eigenvalue problem

$$\begin{gathered}\ell y \equiv (p\,(x)\,y'')'' - (q(x)y')' + r(x)y = \lambda\tau(x)\,y\, + \alpha(x)y^+ + \\ \beta(x)y^- + h(x,y,y',y'',y''',\lambda),\ x \in (0,l),\end{gathered} \tag{1}$$

$$y'(0)\cos\alpha - (py'')(0)\sin\alpha = 0, \tag{2}$$

$$y(0)\cos\beta + Ty(0)\sin\beta = 0, \tag{3}$$

$$y'(l)\cos\gamma + (py'')(l)\sin\gamma = 0, \tag{4}$$

$$y(l)\cos\delta + Ty(l)\sin\delta = 0, \tag{5}$$

where $\lambda \in \mathbb{R}$ is an eigenvalue parameter, $p \in C^2([0,l];(0,+\infty))$, $q \in C^1([0,l];[0,+\infty))$, r, α, $\beta \in C([0,l];\mathbb{R})$, $\tau \in C([0,l];(0,+\infty)$, $y^+(x) = \max\{y(x),0\}$ and $y^-(x) = (-\,y(x))^+$. The function h has the form $f + g$, where $f, g \in C([0,l] \times \mathbb{R}^5;\mathbb{R})$ and satisfy the following conditions: there exist constant $M > 0$ and sufficiently large number $\varrho > 0$ such that

$$\begin{gathered}\left|\frac{f(x,y,s,v,w,\lambda)}{y}\right| \le M,\ x \in [0,l],\ (y,s,v,w) \in \mathbb{R}^4,\ y \neq 0, \\ |y| + |s| + |v| + |w| \ge \varrho,\ \ \lambda \in \mathbb{R};\end{gathered} \tag{6}$$

for any bounded interval $\Lambda \subset \mathbb{R}$,

$$g(x,y,s,v,w,\lambda) = o(|y| + |s| + |v| + |w|) \text{ as } |y| + |s| + |v| + |w| \to \infty, \tag{7}$$

uniformly in $x \in [0,l\,]$ and $\lambda \in \Lambda$.

Global bifurcation of nontrivial solutions of problem (1)-(5) in the case $\alpha = \beta = \gamma = \delta = 0$ was considered in [1].

Let $(b.c.)$ be the set of functions that satisfy the boundary conditions (2)-(5).

By $E = C^3[0,l] \cap (b.c.)$ we denote the Banach space with the usual norm $||u||_3 = ||u||_\infty + ||u'||_\infty + ||u''||_\infty + ||u'''||_\infty$.

Let S_k^ν be the set of functions $y \in E$ that have the oscillation properties of eigenfunctions of the linear problem obtained from (1)-(5) by setting $\alpha \equiv 0$, $\beta \equiv 0$, $h \equiv 0$, and their derivatives (see [2, § 3.1]). Using the method of proof of Theorem 3.3 of the paper [3] we can show that the half-linear eigenvalue problem

$$\begin{cases} (\ell y)(x) = \lambda\tau(x)\, y(x) + \alpha(x) y^+(x) + \beta(x) y^-(x), \ x \in (0,l), \\ y \in (b.c.), \end{cases} \tag{8}$$

which obtained from (1)-(5) by setting $h \equiv 0$ possesses two unbounded sequences of simple half-eigenvalues

$$\lambda_1^+ < \lambda_2^+ < \ldots < \lambda_k^+ < \ldots \ \text{ and } \ \lambda_1^- < \lambda_2^- < \ldots < \lambda_k^- < \ldots .$$

For each $k \in \mathbb{N}$ the half-eigenfunctions y_k^+ and y_k^- corresponding to the half-eigenvalues λ_k^+ and λ_k^- are contained in the sets S_k^+ and S_k^-, respectively. Furthermore, aside from solutions on the collection of the half-lines $\{(\lambda_k^+, ty_k^+) : t > 0\}$ and $\{(\lambda_k^-, ty_k^-) : t > 0\}$, and trivial ones, problem (8) has no other solutions.

Lemma 1. *Let $f \equiv 0$ and condition (7) be satisfied. Then for each $k \in \mathbb{N}$ the points (λ_k^+, ∞) and (λ_k^-, ∞) are asymptotic bifurcation points of problem (1)-(5) with respect to the set $\mathbb{R} \times S_k^+$ and $\mathbb{R} \times S_k^-$, respectively.*

Theorem 1. *Let $f \equiv 0$ and condition (7) be satisfied. For each $k \in \mathbb{N}$ there exist connected components C_k^+ and C_k^- of nontrivial solutions of problem (1)-(5) with $h \equiv 0$ which meet (λ_k^+, ∞) and (λ_k^-, ∞) with respect to the sets $\mathbb{R} \times S_k^+$ and $\mathbb{R} \times S_k^-$, respectively. Moreover, for each $k \in \mathbb{N}$ and each $\nu \in \{+, -\}$ for the set C_k^ν one of the following statements holds: (i) C_k^ν meets $(\lambda_{k'}^{\nu'}, \infty)$ with respect to the set $\mathbb{R} \times S_{k'}^{\nu'}$ for some $(k', \nu') \neq (k, \nu)$; (ii) C_k^ν meets $(\lambda, 0)$ for some $\lambda \in \mathbb{R}$; (iii) the projection of the set C_k^ν on $\mathbb{R} \times \{0\}$ is unbounded.*

We introduce the following notations:

$$N_\alpha = \max_{x \in [0,\,l]} |\alpha(x)|, \ N_\beta = \max_{x \in [0,\,l]} |\beta(x)|, \ N_{\alpha,\beta,M} = N_\alpha + N_\beta + M,$$

and

$$I_k^+ = \left[\lambda_k^+ - \frac{N_{\alpha,\beta,M}}{\tau_0}, \lambda_k^+ + \frac{N_{\alpha,\beta,M}}{\tau_0}\right], \ I_k^- = \left[\lambda_k^- - \frac{N_{\alpha,\beta,M}}{\tau_0}, \lambda_k^- + \frac{N_{\alpha,\beta,M}}{\tau_0}\right],$$

where $\tau_0 = \min\limits_{x\in[0,\,l]} \tau(x)$.

We have also the following results.

Lemma 3. *Let conditions (6) and (7) be satisfied. Then for each* $k \in \mathbb{N}$ *the set of asymptotic bifurcation points of problem (1)-(5) with respect to the set* $\mathbb{R}\times S_k^+$ *and* $\mathbb{R}\times S_k^-$ *are nonempty. In addition, if* (λ, ∞) *is an asymptotic bifurcation point of problem (1)-(5) with respect to the set* $\mathbb{R}\times S_k^+$ *or* $\mathbb{R}\times S_k^-$*, then* $\lambda \in I_k^+$ *or* $\lambda \in I_k^-$.

Theorem 1. *Let conditions (6) and (7) be satisfied. Then for each* $k \in \mathbb{N}$ *there exist connected components* D_k^+ *and* D_k^- *of the set of nontrivial solutions of problem (1)-(5) which meet* $I_k^+\times\{\infty\}$ *and* $I_k^-\times\{\infty\}$*, respectively. Moreover, for each* $k \in \mathbb{N}$ *and each* $\nu \in \{+\,-\}$ *for the set* D_k^ν *at least one of the following statements holds: (i)* D_k^ν *meets* $I_{k'}^{\nu'} \times \{\infty\}$ *with respect to the set* $\mathbb{R}\times S_{k'}^{\nu'}$ *for some* $(k', \nu') \neq (k, \nu)$*; (ii)* D_k^ν *meets* $\mathbb{R}\times\{0\}$ *for some* $\lambda \in \mathbb{R}$*; (iii) the projection of the set* D_k^ν *on* $\mathbb{R}\times\{0\}$ *is unbounded.*

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On the uniqueness of inverse problems for 'Weighted' Sturm-Liouville operator with δ–interaction

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In this study, we consider the following Sturm-Liouville problem $L := L(q(x))$

$$\ell[y] := -y'' + q(x)y = \lambda y, \quad x \in \left(0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right), \tag{1}$$

with the boundary conditions

$$U(y) := y'(0) = 0, \quad V(y) := y(\pi) = 0, \tag{2}$$

and conditions at the points $x = \pi/2$,

$$I(y) := \left\{ \begin{array}{c} y\left(\frac{\pi}{2}+0\right) = y\left(\frac{\pi}{2}-0\right) \equiv y\left(\frac{\pi}{2}\right) \\ y'\left(\frac{\pi}{2}+0\right) - y'\left(\frac{\pi}{2}-0\right) = -\alpha\lambda y\left(\frac{\pi}{2}\right), \end{array} \right. \tag{3}$$

where $q(x)$ is real-value function in $W_2^1(0,\pi)$ and $\alpha > 0$; λ is spectral parameter. Notice that, we can understand problem (1), (3) as studying the equation

$$-y'' + q(x)y = \lambda\rho(x)y, \quad x \in (0,\pi), \tag{4}$$

where $\rho(x) = 1 + \alpha\delta\left(x - \frac{\pi}{2}\right)$, $\delta(x)$ is the Dirac function (see [1]).

After construction of the Hilbert space related to (4), we establish various uniqueness result of inverse problems [2] for operator L.

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Some necessary conditions for an extremum in variational problems with delay

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In the present paper, we consider the following vector problem of variation with delayed argument:

$$S(x(\cdot)) = \int_{t_0}^{t_1} L(t, x(t), x(t-h), \dot{x}(t), \dot{x}(t-h))dt \to \underset{x(\cdot)}{ext}, \tag{1}$$

$$x(t) = f(t), t \in [t_0 - h, t_0], x(t_1) = x_1, x_1 \in R^n, \tag{2}$$

here R^n is n-dimensional space, $t_0, t_1 \in (-\infty, +\infty), x_0$ and x_1 are the given points $h = const > 0$, $t_1 - t_0 > h$, functions $L(t, x, y, \dot{x}, \dot{y}) : [t_0, t_1] \times R^n \times R^n \times R^n \times R^n \to R := (-\infty, +\infty)$ and $f(t) \in C^1([t_0 - h, t_0], R^n)$ are the given continuously differentiable functions with respect to all their arguments and for each $(t, x, y, \dot{x}, \dot{y}) \in (t_1, +\infty) \times R^n \times R^n \times R^n \times R^n$ there is equality $L(t, x, y, \dot{x}, \dot{y}) = 0$, where $y = y(t) = x(t-h)$, $\dot{y} := \dot{y}(t) = \dot{x}(t-h)$, $t \in I := [t_0, t_1]$, moreover $x(t) \in KC^1\left(\hat{I}, R^n\right)$, where $\hat{I} := [t_0 - h, t_1]$ and $KC^1\left(\hat{I}, R^n\right)$ is a class of piecewise-smooth function.

We call the functions $x(\cdot) \in KC^1\left(\hat{I}, R^n\right)$ satisfying boundary conditions (2), an admissible function.

It should be noted that problem (1), (2) was studied in works [1-4] and in these works the analogs of the Euler equation, Weierstrass and Legendre necessary conditions were obtained.

The following theorems are proved.

Theorem 1. Let the admissible function $\overline{x}(\cdot)$ provide at least a weak extremum for the functional (1). Then, there exists a constant vector c, such that $\overline{x}(\cdot)$ satisfies the equation:

$$\overline{L}_{\dot{x}}(t) + \overline{L}_{\dot{y}}(t+h) = \int_{t_0}^{t} \left[\overline{L}_x(s) + \overline{L}_y(s+h)\right] ds + c,$$

$$t \in [t_0, t_1] \setminus \{t\}, \tag{3}$$

$$\overline{x}(t) = f(t), t \in [t_0 - h, t_0].$$

Here $\overline{L}_{\dot{x}}(t) := L_{\dot{x}}\left(t, \overline{x}(t), \overline{y}(t), \dot{\overline{x}}(t), \dot{\overline{y}}(t)\right)$ (the symbols $\overline{L}_x(\cdot)$, $\overline{L}_y(\cdot)$, $\overline{L}_{\dot{y}}(\cdot)$ are defined similarly and the symbol $\{t\}$ is a set of corner points of the function $\overline{x}(\cdot)$.

Theorem 2. Let an admissible function $\overline{x}(\cdot)$ be a strong local extremum in the problem (1),(2). Then:

(i) if $\theta \in (t_0, t)$ is a corner point of the function $\overline{x}(\cdot)$, then the equality holds:

$$\overline{L}_{\dot{x}}(t) + \overline{L}_{\dot{y}}(t+h) \Big|_{\substack{t=\theta+0 \\ t=\theta-0}} = 0, \tag{4}$$

(ii) if $\theta \in (t_0, t_1)$ is a corner point of the function $\overline{x}(\cdot)$, but the points $\theta + h, \theta - h$ are not corner points of the function $\overline{x}(\cdot)$, then the equality holds:

$$\overline{L}(t) + \overline{L}(t+h) - \dot{x}(t)\left[\overline{L}_{\dot{x}}(t) + \overline{L}_{\dot{y}}(t+h)\right] \Big|_{\substack{t=\theta+0 \\ t=\theta-0}} = 0. \tag{5}$$

Note that the statement (4) is true for weak local extremums as well.

By virtue of Theorem 1, we assert that if the admissible function $\overline{x}(\cdot)$ is a weak local extremum in problem (1),(2), then it is a solution of the integro-differential equations (3). Statements (4), (5) of Theorem 2 are analogs of the Weierstrass-Erdmann conditions for the problems (1), (2).

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Studying the Goursat-Darboux problem with integral conditions

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Let the function $y = y(t, x)$ be determined as the solution of the equation

$$y_{tx} = f(t, x, y, y_z, y_x), \quad (t, x)\varphi \tag{1}$$

under the constraints

$$\begin{cases} \int\limits_0^T \chi(t)y(t,x)dt = \varphi(x), & x \in [0, l], \\ \int\limits_0^l m(x)y(t,x)dt = \varphi(x), & t \in [0, T], \end{cases} \tag{2}$$

where $Q = \{(t,x) : 0 \le t \le T,\ 0 \le x \le l\}, T, l > 0$ are the given numbers $y = (y_1, y_2, ..., y_n)^T$, is a state vector $f = (f_1, f_2, ..., f_n)$, $\varphi = (\varphi_1, \varphi_2, ..., \varphi_n)^T$ $\psi = (\psi_1, \psi_2, ..., \psi_n)^T$ are vector-function ; $x(t)$ and $m(x)$ are the prescribed matrices of dimension $n \times m$.

We will assume that:

a) the function $f(t, x, y, p, q)$ is continuous in totality of its arguments:

b) the matrices $\chi(t) \in L_\infty[0,T]$, $m(x) \in L_\infty[0,l]$ are perumutational , i.e. $\chi(t)m(t) = m(x)\chi(t)$ for $(t,x) \in Q$ and $\det\left[\int\limits_0^T \chi(t)dt\right] \neq 0$, $\det\left[\int\limits_0^l m(x)dx\right] \neq 0$.

c) the functions $\varphi(x)$ and $\psi(t)$ are differentiable on $[0,l]$ and $[0,T]$, respectively and furthermore, the following matching conditions are fulfilled:

$$\int_0^l m(x)\varphi(x)dt = \int_0^T \chi(t)\psi(t)dt \equiv A = const.$$

Theorem. *The problem (1)-(2) is equivalent to the following integral equation*

$$\begin{aligned} y(t,x) &= \widetilde{m}^{-1}(l)\psi(t) + \widetilde{\chi}^{-1}(T)\varphi(x) - \widetilde{\chi}^{-1}(T)\widetilde{m}^{-1}(l)A + \\ &+ \int_0^T \int_0^l G(t,x,\tau,s) f(t,x,y,y_z,y_x)d\tau ds, \end{aligned}$$

where

$$G(t,x,\tau,s) = \begin{cases} \widetilde{m}^{-1}(l)\widetilde{\chi}^{-1}(T)\int\limits_0^s m(r)dr\int\limits_0^\tau x(\alpha)d\alpha, & 0 \le \tau \le t, \quad 0 \le s \le x, \\ -\widetilde{m}^{-1}(l)\widetilde{\chi}^{-1}(T)\int\limits_0^s m(r)dr\int\limits_0^\tau x(\alpha)d\alpha, & 0 \le s \le x, \quad \tau \le t \le T, \\ -\widetilde{m}^{-1}(l)\widetilde{\chi}^{-1}(T)\int\limits_0^\tau x(\alpha)d\alpha\int\limits_s^l m(r)dr, & 0 \le \tau \le t \quad s < x \le l, \\ \widetilde{m}^{-1}(l)\widetilde{\chi}^{-1}(T)\int\limits_s^s m(r)dr\int\limits_\tau^s x(\alpha)d\alpha, & x < s \le l, \quad t < \tau \le T, \end{cases}$$

$$\widetilde{m}^{-1}(l) = \left(\int_0^l m(x)dx\right)^{-1},\ \widetilde{\chi}^{-1}(T) = \left(\int_0^T \chi(t)dt\right)^{-1}.$$

The problem (1), (2) was also studied in the paper [1]

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On the existence and uniqueness of the solution to a mixed problem for one type of non-classical equations

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We study the solvability of the mixed problem

$$M\left(t, \frac{\partial}{\partial t}\right)U = L\left(x, \frac{\partial}{\partial x}\right)U, 0 < t < T,\ \ 0 < x < 1 \tag{1}$$

$$U(0,x) = \varphi(x) \tag{2}$$

$$U(t,0) = U(t,1) = 0 \tag{3}$$

where $M\left(t, \frac{\partial}{\partial t}\right) = \frac{1}{P(t)}\frac{\partial}{\partial t}$, $L\left(x, \frac{\partial}{\partial x}\right) = \frac{1}{(x+b)^2} \cdot \frac{\partial^2}{\partial x^2}$, $b = b_1 + ib_2$, $P\left(t\right) = p_1\left(t\right) + ip_2\left(t\right)$, are complex-valued functions, $p_j\left(t\right) \in C[0,\,1]$ $(j = 1,\,2)$, $p_1\left(t\right) \neq 0$, $\varphi\left(x\right)$ is a prescribed, $U\left(x\right)$ is a desired function

$$1^0. \int_0^t p_1\left(\tau\right)d\tau\ > 0,\ b_1 < -1\ ,\ b_2 > 0, ;$$

2^0. $Re\,(1+b)^2 + \omega\,(0)\,Im\,(1+b)^2 > 0$, if $Im\left[\overline{p}\cdot \int_0^t p\,(\tau)\,d\tau\right] \geq 0$ and $Re\,(1+b)^2 + \omega\,(T)\,Im\,(1+b)^2 > 0$ if $Im\left[\overline{p}\cdot \int_0^t p\,(\tau)\,d\tau\right] < 0$ where

$$\omega\,(t) = \int_0^t p_2\,(\tau)\,d\tau \cdot \left(\int_0^t p_1\,(\tau)\,d\tau\right)^{-1}$$

$$3^0.\varphi(x) \in C^2[0,\ 1], \varphi(0) = \varphi(1) = 0$$

Are final conditions for solvability.

It is easy to see that even subject to inequalities $p_1\,(t) > 0$, $b_1 < -1$, $b_2 > 0$ equation (1) is I.G.Petrovsky iff

$$Im\,[\overline{p}\cdot\,(p'\,(t))] \leq 0,\;\; Re\,(1+b)^2 + r\,(0)\,Im\,(1+b)^2 > 0 \tag{4}$$

or

$$Im\,[\overline{p}\cdot\,(p'\,(t))] > 0,\;\; Re\,(1+b)^2 + r\,(T)\,Im\,(1+b)^2 > 0 \tag{5}$$

where $r\,(t) = p_2\,(t)\;(p_1\,(t))^{-1}$.

The following theorem is valid.

Theorem. Let conditions 1^0, $2^0, 3^0$ be fulfilled. Then problem (1)-(3) has the classic solution $U\,(t,\,x) \in C^{1,2}\,((0;T]\times[0;1])\bigcap C\,([0;T]\times[0;1])$ represented by the following formula for $t > 0$

$$U\,(t,\,x) = \frac{1}{\pi i}\int \lambda e^{\lambda^2 \int\limits_0^t P(\tau)d\tau}\cdot y\,(x,\lambda)\,d\lambda$$

where

$$\Gamma = \bigcup_{j=1}^{3}\Gamma_j$$

$$\Gamma_j = \{\lambda :\ \lambda = r\,(1+p_j)\,,\, r \geq R\}\;\;(j=1,2)$$

$$\Gamma_3 = \{\lambda :\ \lambda = R\,(1+i\eta)\,,\, p_1 \leq \eta \leq p_2\}\;\;,$$

$$y\,(x,\lambda) = \int_0^1 G\,(x,\,\xi,\lambda)\,(\xi+b)^2\,\varphi\,(\xi)\,d\xi$$

$$p_j = K_j\,(t_j) + (-1)^j\,\delta,\;\; K_j\,(t_j) = -\omega\,(t) + (-1)^j\sqrt{\omega^2\,(t)+1},\;\;(j=1,2) \tag{6}$$

$$\omega(t) = \int_0^t p_2(\tau)\,d\tau \cdot \left(\int_0^t p_1(\tau)\,d\tau\right)^{-1}$$

$t_1 = 0,\ t_2 = T$ if $Im\left[\overline{p} \cdot \int_0^t p(\tau)\,d\tau\right] \geq 0$ and $t_1 = T,\ t_2 = 0$ if $Im\left[\overline{p} \cdot \int_0^t p(\tau)\,d\tau\right] < 0$, R is a rather large number, δ is a rather small positive number .

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Studying an eigen-value problem

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In the paper, we consider a general form boundary value problem stated for a variable coefficient second-order differential equation. Having written the expression of the solution with ceratin conditions imposed on the coefficient of the equation and the boundary conditions, the asymptotic representation of eigen-values is found in certain cases.

In the paper, we consider the problem of finding the solution of the equation

$$y'' - \lambda^2 a(x)y = \varphi(x),\ 0 < x < 1 \tag{1}$$

satisfying the conditions

$$l_1(y) = \alpha_{11}y(0) + \alpha_{12}y'(0) + \beta_{11}y(1) + \beta_{12}y'(1) = 0 \tag{2}$$

$$l_2(y) = \alpha_{21}y(0) + \alpha_{22}y'(0) + \beta_{21}y(1) + \beta_{22}y'(1) = 0$$

Here $a(x)$ is a real variable real valued function, $a(x) > 0$, $a\,(x) \in C[0,\,1]$, α_{ij}, β_{ij}, $(i, j = 1,\,2)$ are complex numbers.

Assume that the following conditions are fulfilled.

$$1^0.a\,(x) > 0, a\,(x) \in C[0,\,1]$$

$$2^0.\varphi(x) \in C^2[0,\ 1], \varphi(0) = \varphi(1) = 0.$$

$$3^0.\begin{vmatrix} \alpha_{12} & \beta_{12} \\ \alpha_{22} & \beta_{22} \end{vmatrix} \neq 0$$

It is known that [2], [4] the asymptotic representation of the system of fundamental solutions of a homogeneous equation corresponding to the equation (1) is in the form:

$$\frac{d^j y_k\,(x,\,\lambda)}{dx^j} = \left((-1)^{k-1}\cdot\lambda\right)^j \cdot \left[1 + \frac{E_{j,k}\,(x,\,\lambda)}{\lambda}\right]\cdot e^{(-1)^k\cdot\lambda\cdot\int_0^x \sqrt{a(\eta)}d\eta} \qquad (|\lambda| \to \infty)$$

$$(j = 0,\,1;\ \ k = 1, 2;\ \lambda \in S_i;\ i = 1, 2)$$

Here the functions $E_{j,k}\,(x,\,\lambda)$ are bounded continuous functions for $\lambda \in S_i = \left\{\lambda\backslash\ (-1)^i\,Re\lambda < 0\right\}; i = 1, 2;\ x \in [0,\ 1]$

The following theorems are proved.

Theorem 1. Assume that conditions 1^0, 3^0 are satisfied. The eigen-values of the spectral problem (1)-(2) have the following form asymptotic representation:

$$\lambda_k = \frac{\pi k\sqrt{-1}}{\int_0^1 \sqrt{a\,(\eta)}d\eta} + O\left(\frac{1}{k}\right),\ \ (|k| \to \infty)$$

Theorem 2. Assume that conditions 1^0, 2^0, 3^0 are satisfied. Then the spectral problem (1)-(2) has a solution in the form

$$y(x,\lambda) = \int_0^1 G(x,\xi,\lambda)\,\varphi(\xi)\,d\xi,$$

$$y(x,\lambda) = \int_0^1 G(x,\xi,\lambda)\,\varphi(\xi)\,d\xi.$$

Here $G(x,\xi,\lambda) = \frac{\Delta(x,\xi,\lambda)}{\Delta(\lambda)}$,

$$\Delta(\lambda) = \begin{vmatrix} A_{11}(\lambda) + B_{11}(\lambda)\, e^{\lambda \int_0^1 \sqrt{a(\eta) d\eta}} & A_{12}(\lambda) + B_{12}(\lambda)\, e^{-\lambda \int_0^1 \sqrt{a(\eta) d\eta}} \\ A_{21}(\lambda) + B_{21}(\lambda)\, e^{\lambda \int_0^1 \sqrt{a(\eta) d\eta}} & A_{22}(\lambda) + B_{22}(\lambda)\, e^{-\lambda \int_0^1 \sqrt{a(\eta) d\eta}} \end{vmatrix},$$

$$\Delta(x,\xi,\lambda) = \begin{vmatrix} g(x,\xi,\lambda) & y_1(x,\lambda) & y_2(x,\lambda) \\ l_1(g(x,\xi,\lambda))_x & l_1(y_1(x,\lambda)) & l_1(y_2(x,\lambda)) \\ l_2(g(x,\xi,\lambda))_x & l_2(y_1(x,\lambda)) & l_2(y_2(x,\lambda)) \end{vmatrix},$$

$$l_i(y_j(x,\lambda)) = A_{ij}(\lambda) + B_{ij}(\lambda)\, e^{(-1)^{j+1}\lambda \int_0^1 \sqrt{a(\eta) d\eta}}, \ i,j = 1,2,$$

$$g(x,\xi,\lambda) = \pm \frac{\begin{vmatrix} y_1(\xi,\lambda) & y_2(\xi,\lambda) \\ y_1(x,\lambda) & y_1(x,\lambda) \end{vmatrix}}{2w(\xi,\lambda)},$$

$$y_j(x,\lambda) = \left(1 + \frac{E_{1j}(x,\lambda)}{\lambda}\right) e^{(-1)^{1+j}\lambda \int_0^x \sqrt{a(\eta) d\eta}}, \ j = 1, 2$$

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Basis property of root functions of an eigenvalue problem describing bending vibrations of a beam at the ends of which load and inertial load are concentrated

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We consider the following eigenvalue problem

$$y^{(4)}(x) - (q(x)y'(x))' = \lambda y(x), \ 0 < x < 1, \tag{1}$$

$$y''(0) = 0, \tag{2}$$

$$Ty(0) - a\lambda y(0) = 0, \tag{3}$$

$$y''(1) - b\lambda y'(1) = 0, \tag{4}$$

$$Ty(1) - c\lambda y(1) = 0, \tag{5}$$

where $\lambda \in \mathbb{C}$ is a spectral parameter, q is a positive absolutely continuous function on $[0, 1]$, a, b and c are real constants such that

$$a < 0, \ b > 0, \ c > 0.$$

This problem describes small bending vibrations of a homogeneous Euler-Bernoulli beam of constant rigidity, in the cross sections of which a longitudinal force acts, at the left end of which a load is concentrated, and at the right end an inertial load is concentrated (see [1, p. 152-154]).

Problem (1)-(5) in the case of $a > 0$, $b > 0$ and $c < 0$ was considered in [2], where this problem reduces to the spectral problem for a self-adjoint operator in the Hilbert space $H = L_2(0,1) \oplus \mathbb{C}^3$ with the corresponding scalar product. There it was shown that in this case, the eigenvalues of this operator are positive and simple.

In the case of $a < 0$, $b > 0$ and $c > 0$ problem (1)-(5) reduces to the eigenvalue problem for the J-self-adjoint operator in the Pontryagin space $\Pi_2 = L_2(0,1) \oplus \mathbb{C}^3$ with the corresponding inner product, and consequently, can have non-real or multiple real eigenvalues.

In [3] it was shown that the eigenvalues λ_k, $k = 1, 2, \ldots$, of the problem, are real, and all nonzero eigenvalues are simple; in the case $c < 1$ and $a = c-1$, the eigenvalue $\lambda = 0$ is double, and in the cases of $c < 1$, $a \neq c-1$ and $c \geq 1$, the eigenvalue $\lambda = 0$ is simple. Moreover, the eigenvalues have the following location on the real axis:

$$\lambda_2 < \lambda_2 < 0 = \lambda_3 < \lambda_4 < \ldots \text{ if } c < 1 \text{ and } a > c-1,$$

$$\lambda_1 < \lambda_2 = 0 = \lambda_3 < \lambda_4 < \ldots \text{ if } c < 1 \text{ and } a = c-1,$$

$$\lambda_1 < \lambda_2 = 0 < \lambda_3 < \lambda_4 < \ldots \text{ if } c < 1 \text{ and } a < c-1 \text{ or } c \geq 1.$$

Let $\{y_k\}_{k=1}^{\infty}$ be the system of root functions corresponding to the system of eigenvalues $\{\lambda_k\}_{k=1}^{\infty}$ of the problem (1)-(3).

Let i, j, r, $i < j < r$, be arbitrary fixed natural numbers. Now we demonstrate sufficient conditions for the system $\{y_k(x)\}_{k=1,\, k\neq i,\, j,\, r}^{\infty}$ of root functions of problem (1)-(3) to form a basis in $L_p(0,1)$, $1 < p < \infty$.

The main result of this note is the following theorem.

Theorem 1. *Let i, j, r, $i < j < r$, be arbitrary fixed natural numbers. Then the system $\{y_k(x)\}_{k=1,\, k\neq i,\, j,\, l}^{\infty}$ forms a basis in $L_p(0,1)$, $1 < p < \infty$, which is an unconditional basis for $p = 2$, in the following cases:*

(i) i, j, r are sufficiently large and any two of them have the same parity, and the third has the opposite parity;

(ii) $i = 2$ for $c < 1$ and $a \geq c-1$, $i = 3$ for $c < 1$ and $a < c-1$, or $c \geq 1$, j, r are sufficiently large and one of the following conditions hold:

$(ii)_1$ j and r are even, and $c \neq |a|\sqrt{2}$;

$(ii)_2$ r is odd;

$(ii)_3$ j is odd and r is even, and $c \geq |a|\sqrt{2}$ or $c < |a|\sqrt{2}$ and $\frac{r-4/11}{j-4/11} \neq \frac{|a|\sqrt{2}+c}{|a|\sqrt{2}-c}$.

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On the classical solvability of a three-dimensional inverse hyperbolic problem with time-dependent coefficient

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This paper investigates the problem of determining the solution and unknown time-dependent coefficient in a three-dimensional hyperbolic equation with an integral overdetermination condition. First, the considered problem is reduced to an auxiliary problem and the equivalence of the auxiliary problem to the original problem is proved. Further, using the Fourier method, the auxiliary problem is represented in the form of a system of integral equations, and for sufficiently small values of time, the unique existence of a solution of the resulting system in the special functional spaces is sown. In conclusion, using the equivalence of these problems, the existence and uniqueness theorem for the classical solution of the original inverse problem is proved (cf.[1]–[3]).

Mathematical formulation of the problem. Let T be a given positive number and let $D = Q_{xyz} \times \{0 \le t \le T\}$, where Q_{xyz} is defined by the inequalities $0 < x < 1,\ 0 < y < 1$, and $0 < z < 1$. Moreover, we assign $D_T := \bar{D}$ and consider the problem of recovering the functions $u(x,y,z,t) \in C^2(D_T)$ and $a(t) \in C[0,T]$ from the following three-dimensional problem

$$u_{tt}(x,y,z,t) = u_{xx}(x,y,z,t) + u_{yy}(x,y,z,t) + u_{zz}(x,y,z,t)$$

$$+a(t)u(x,y,z,t) + f(x,y,y,t) \ \ (x,y,z,t) \in D_T, \tag{1}$$

$$u(x,y,z,0) = \varphi(x,y,z),\ u_t(x,y,z,0) = \psi(x,y,z),\ x,y,z \in [0,1], \tag{2}$$

$$u(0,y,z,t) = u_x(1,y,z,t) = 0,\ y,z \in [0,1],\ t \in [0,T], \tag{3}$$

$$u(x,0,z,t) = u_y(x,1,z,t) = 0,\ x,z \in [0,1],\ t \in [0,T], \tag{4}$$

$$u(x,y,0,t) = u_z(x,y,1,t) = 0,\ x,y \in [0,1],\ t \in [0,T], \tag{5}$$

$$u(1,1,1,t) + \int_0^1\int_0^1\int_0^1 w(x,y,z)u(x,y,z,t)dxdydz = h(t),\ t \in [0,T], \tag{6}$$

where $f(x,y,z,t), \varphi(x,y,z), \psi(x,y,z), w(x,y,z)$, and $h(t)$ are given functions.

The following theorem is valid.

Theorem 1. *Assume that $\varphi(x,y,z),\ \psi(x,y,z),\ w(x,y,z) \in C(Q_{xyz})$, $f(x,y,z,t) \in C(D_T)$, $h(t) \in C^2[0,T]$, $h(t) \ne 0,\ t \in [0,T]$, and the compatibility conditions*

$$\begin{gathered}
\varphi(1,1,1) + \int_0^1\int_0^1\int_0^1 w(x,y,z)\varphi(x,y,z)dxdydz = h(0), \\
\psi(1,1,1) + \int_0^1\int_0^1\int_0^1 w(x,y,z)\psi(x,y,z)dxdydz = h'(0),
\end{gathered} \tag{7}$$

hold. Then the problem of finding a classical solution of (1)-(6) is equivalent to the problem of determining the functions $u(x,y,z,t) \in C^2(D_T)$ and $a(t) \in C[0,T]$ satisfying (1)-(5), and the condition

$$h''(t) = u_{xx}(1,1,1,t) + u_{yy}(1,1,1,t) + u_{zz}(1,1,1,t)$$

$$+\int_0^1\int_0^1\int_0^1 w(x,y,z)(u_{xx}(x,y,z,t)+u_{yy}(x,y,z,t)+u_{zz}(x,y,z,t))dxdydz$$

$$+a(t)h(t)+f(1,1,1,t)+\int_0^1\int_0^1\int_0^1 w(x,y,z)f(x,y,z,t)dxdydz,\ t\in[0,T]. \quad (8)$$

Existence and uniqueness of the classical solution. Let us impose the following conditions on the functions f,φ,ψ,w, and h:

$C_1)$ $\varphi(x,y,z)\in C^2(\bar{Q}_{xy,z}),\varphi_{xxy}(x,y,z),\varphi_{xyy}(x,y,z),\varphi_{xxx}(x,y,z),\varphi_{yyy}(x,y,z),$
$\varphi_{xyz}(x,y,z),\varphi_{xxz}(x,y,z),\varphi_{xzz}(x,y,z),\varphi_{yzz}(x,y,z),\varphi_{yyz}(x,y,z),$
$\varphi_{zzzy}(x,y,z)\in L_2(Q_{xy}),$
$\varphi(0,y,z)=\varphi_x(1,y,z)=\varphi_{xx}(0,y,z)=0,\quad y,z\in[0,1],$
$\varphi(x,0,z)=\varphi_y(x,1,z)=\varphi_{yy}(x,0,z)=0,\quad x,z\in[0,1],$
$\varphi(x,y,0)=\varphi_z(x,y,1)=\varphi_{zz}(x,y,0)=0,\quad x,y\in[0,1];$

$C_2)$ $\psi(x,y,z)\in C^1(\bar{Q}_{xy}),\psi_{xx}(x,y,z),\psi_{yy}(x,y,z),\psi_{zz}(x,y,z)\in L_2(Q_{xyz}),$
$\psi(0,y,z)=\psi_x(1,y,z)=0,\quad y,z\in[0,1],$
$\psi(x,0,z)=\psi_y(x,1,z)=0,\quad x,z\in[0,1],$
$\psi(x,y,0)=\psi_z(x,y,1)=0,\quad x,y\in[0,1];$

$C_3)$ $f(x,y,z,t),f_x(x,y,z,t),f_y(x,y,z,t)f_z(x,y,z,t)\in C(D_T),$
$f_{xy}(x,y,z,t),f_{xx}(x,y,z,t),f_{yy}(x,y,z,t)\in L_2(D_T),$
$f(0,y,z,t)=f_x(1,y,z,t)=0,\quad y,z\in[0,1],\quad t\in[0,T],$
$f(x,0,z,t)=f_y(x,1,z,t)=0,\quad x,z\in[0,1],\quad t\in[0,T],$
$f(x,y,0,t)=f_z(x,y,1,t)=0,\quad x,y\in[0,1],\quad t\in[0,T];$

$C_4)$ $w(x,y,z)\in C(\bar{Q}_{xyz}),\ h(t)\in C^2[0,T],\ h(t)\neq 0,\ t\in[0,T].$

Then it is not difficult to prove the following theorem.

Theorem 2. *Let the conditions $C_1)$ - $C_5)$ be satisfied. Then, problem (1)–(5), (8) for the small values of time has a unique solution.*

Thus, the following theorem is easily deduced from Theorem 1 and Theorem 2.

Theorem 3. *Suppose that all the conditions of Theorem 2 and the compatibility conditions (7) are fulfilled. Then the inverse boundary value problem (1)-(6) has a unique classical solution for sufficiently small values of T.*

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Special approximation method for solving system of ordinary differential equations

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Abstract. In this paper, a special approximate method is presented to solve the system of ordinary differential equations. In this method, the given problem is first converted into an integral equation that includes Volterra and Fredholm parts. Then special successive approximation method are performed in the Volterra part. Due to the appearance of the factorial factor in the denominator of its kernel, the Volterra part tends to zero in the next iterations. This causes us to discard Volterra's sentence as an error of method. Finally, the analytical-approximate solution of the problem is obtained by solving the resulting equation, which is the second type of Fredholm integral equation.

Statement of the problem.
Consider the first-order matrix form system of ordinary differential equations as follows:

$$\dot{X}(t) = A(t)X(t) + F(t), \quad t \in (a, b), \tag{1}$$

$$\alpha X(a) + \beta X(b) = \gamma, \tag{2}$$

with general linear boundary conditions. Where X(t) is an unknown vector function, $A(t)$ is a $n \times n$ square matrix with elements of continuous functions, $F(t)$ is a known continuous vector function in the interval $[a, b]$, α and β are real constant matrices of order n, and γ is a column vector with constant values. In addition, the general linear boundary conditions (2) is linearly independent. First, we convert the problem (1)-(2) into an integral equation by integrating the sides in the interval $[a, t]$ as follows [1, 2, 3, 4]:

$$X(t) = X(a) + \int_a^t A(\tau)X(\tau)\,\mathrm{d}\tau + \int_a^t F(\tau)\,\mathrm{d}\tau. \tag{3}$$

$$X(t) = \int_a^t A(\tau)X(\tau)\,\mathrm{d}\tau + f(t) + C, \tag{4}$$

where

$$f(t) = (\alpha + \beta)^{-1}\gamma - \int_a^b (\alpha + \beta)^{-1}\beta F(\tau)\,\mathrm{d}\tau + \int_a^t F(\tau)\,\mathrm{d}\tau,$$

$$C = \int_a^b B(\tau)X(\tau)\,\mathrm{d}\tau, \quad B(\tau) = -(\alpha + \beta)^{-1}\,\beta A(\tau).$$

Now in the integral equation (4), we use the special method of successive approximations only in the Volterra part of the integral equation, we have:

$$X(\tau) = \int_a^\tau A(\tau_1)X(\tau_1)\,\mathrm{d}\tau_1 + f(\tau) + C,$$

which by putting in the integral equation (4) we have: By replacing the above value in the relation (4) and categorizing the sentences relative to C, we have the following relation:

$$A_k = \int_a^t \frac{\left(\int_\tau^t A(\xi)\,\mathrm{d}\xi\right)^k}{k!} f(\tau)\,\mathrm{d}\tau\ , \quad B_k = \int_a^b B(t)\,\mathrm{d}t \int_a^t \frac{\left(\int_\tau^t A(\xi)\,\mathrm{d}\xi\right)^k}{k!}\,\mathrm{d}\tau,$$

the final solution can be summarized as follows

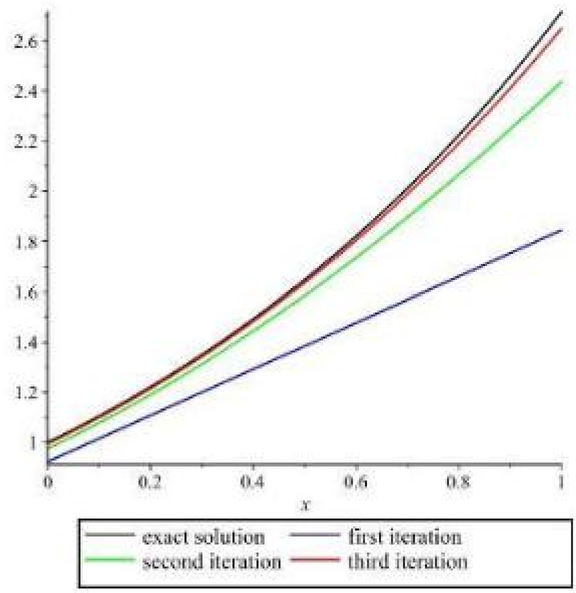
$$X(t) \simeq \Big[I - \sum_{k=1}^{n-1} \int_a^t \frac{\left(\int_\tau^t A(\xi)\,\mathrm{d}\xi\right)^k}{k!}\,\mathrm{d}\tau\Big]\Big[I + \sum_{k=1}^{n-1} B_k - \int_a^b B(t)\,\mathrm{d}t\Big]^{-1}$$

$$\Big[-\sum_{k=1}^{n-1}\int_a^b B(t)\,\mathrm{d}t A_k + \int_a^b B(t)f(t)\,\mathrm{d}t\Big] - \sum_{k=1}^{n-1} A_k + f(t). \tag{5}$$

Example. We consider the following integral equation including Volterra and Fredholm:

$$u(x) = 1 + \int_0^x u(t)\,\mathrm{d}t + \int_0^1 (t^2 + t - 1)u(t)\,\mathrm{d}t, \tag{6}$$

whose analytical answer is $u(t) = e^x$. Now, we solve the integral equation (6) by using the method of successive special approximations, for example, the third time, we see the results in the chart below.



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Boundedness of parabolic fractional integral operators with rough kernels in parabolic local generalized Morrey spaces

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Let P be a real $n \times n$ matrix, whose all eigenvalues have a positive real part, $A_t = t^P$, $t > 0$, $\gamma = trP$ is the homogeneous dimension on R^n and Ω is an A_t- homogeneous of degree zero function, integrable to a power $s > 1$ on the unit sphere generated by corresponding parabolic metric. We find conditions on the pair (φ_1, φ_2) for the boundedness of the operator $I^P_{\Omega,\alpha} f$ from the space $LM^{\{x_0\}}_{p,\varphi_1,P}(R^n)$ to another one $LM^{\{x_0\}}_{q,\varphi_2,P}(R^n)$, $1 < p < q < \infty$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{\gamma}$, and from the $LM^{\{x_0\}}_{1,\varphi_1,P}(R^n)$ to the weak space $WLM^{\{x_0\}}_{p,\varphi_2,P}(R^n)$, $1 \le q < \infty$, $1 - \frac{1}{q} = \frac{\alpha}{\gamma}$.

Definition 1. Let $\varphi(x,r)$ be a positive measurable function on $R^n \times (0,\infty)$ and $1 \le p < \infty$. For any fixed $x_0 \in R^n$ we denote by $LM^{\{x_0\}}_{p,\varphi,P} \equiv LM^{\{x_0\}}_{p,\varphi,P}(R^n)$ the parabolic generalized local Morrey space, the space of all functions $f \in L^{loc}_p(R^n)$ with finite quasinorm

$$\|f\|_{LM^{\{x_0\}}_{p,\varphi,P}} = \|f(x_0+\cdot)\|_{LM_{p,\varphi,P}}.$$

Also by $WLM^{\{x_0\}}_{p,\varphi,P} \equiv WLM^{\{x_0\}}_{p,\varphi,P}(R^n)$ we denote the weak parabolic generalized local Morrey space of all functions $f \in WL^{loc}_p(R^n)$ for which

$$\|f\|_{WLM^{\{x_0\}}_{p,\varphi,P}} = \|f(x_0+\cdot)\|_{WLM_{p,\varphi,P}} < \infty.$$

Definition 2. Let $S_\rho = \{w \in R^n : \rho(w) = 1\}$ be the unit ρ-sphere (ellipsoid) in R^n $(n \ge 2)$ equipped with the normalized Lebesgue surface measure $d\sigma$ and Ω be A_t-homogeneous of degree zero, i.e. $\Omega(A_t x) \equiv \Omega(x)$, $x \in R^n$, $t > 0$. The parabolic fractional integral operator $I^P_{\Omega,\alpha} f$ with rough kernels, $0 < \alpha < \gamma$, of a function $f \in L^{loc}_1(R^n)$ is defined by

$$I^P_{\Omega,\alpha} f = \int_{R^n} \frac{\Omega(x-y)\, f(y)}{\rho(x-y)^{\gamma-\alpha}} dy.$$

We prove the boundedness of the parabolic integral operator $I^P_{\Omega,\alpha}$ with rough kernel from one parabolic local generalized Morrey space $LM^{\{x_0\}}_{p,\varphi_1,P}(R^n)$ to another one $LM^{\{x_0\}}_{q,\varphi_2,P}(R^n)$, $1 < p < q < \infty$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{\gamma}$, and from the space $LM^{\{x_0\}}_{1,\varphi_1,P}(R^n)$ to the weak space $WLM^{\{x_0\}}_{q,\varphi_2,P}(R^n)$, $1 \le q < \infty$, $1 - \frac{1}{q} = \frac{\alpha}{\gamma}$.

Theorem. Suppose that $x_0 \in R^n$, $0 < \alpha < \gamma$ and the function $\Omega \in L_{\frac{\gamma}{\gamma-\alpha}}(S_\rho)$ is A_t-homogeneous of degree zero. Let $1 \le p < \frac{\gamma}{\alpha}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{\gamma}$, and the pair (φ_1, φ_2) satisfy the condition

$$\int_r^\infty \frac{ess\sup\limits_{t<\tau<\infty} \varphi_1(x_0,\tau)\, \tau^{\frac{n}{p}}}{t^{\frac{\gamma}{q}+1}} dt \le C \varphi_2(x_0,r),$$

where C does not depend on x_0 and r. Then the operator $I^P_{\Omega,\alpha}$ is bounded from $LM^{\{x_0\}}_{p,\varphi_1,P}$ to $LM^{\{x_0\}}_{q,\varphi_2,P}$ for $p > 1$ and from $LM^{\{x_0\}}_{1,\varphi_1,P}$ to $WLM^{\{x_0\}}_{q,\varphi_2,P}$ for $p = 1$.

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Conditions for equiconvergence of expansion in eigenfunctions of one Schrödinger operator with generalized potential

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We consider the operator $L = -\Delta + q(x)$, where $q(x)$ is a generalized function generated by a complex-valued measure σ, for which there exists an integral

$$J(x) = \int_{E_3} \frac{|d(\sigma(y))|}{|x-y|}$$

and holds the inequality $\sup\limits_{x} |J(x)| \le C_1 < \infty$.

Based on work [1] for a finite function

$$f(x) \in D(L) =$$
$$= \left\{ f \in L_2(E_3) \,|\, f \in C(E_3) \bigcap L_2(E_3),\ -\Delta f + qf \in L_2(E_3) \right\}$$

at $Imk > 0$ $\left(k \neq k_i,\, i = \overline{1, p_1},\ \ k_i - \text{eigenvalues}\right)$ occurs

$$\int_{E_3} G(x,y,k) f(y)\, dy = \sum_{j=1}^{p_1} \int_{E_3} A_j(x,y,k) f(y)\, dy +$$
$$\sum_{j=1}^{n_2} \int_{E_3} Y_j(x,y,k) f(y)\, dy + \frac{2}{\pi} \int_{\Gamma} \int_{W} \frac{s^2 \psi(x,\omega,-s)\, \Phi(\omega,s)}{s^2-k^2} ds d\omega, \qquad (1)$$

where $G(x,y,\lambda)$ - kernel of the resolvent of the operator L.

Applying the operator $(L-k^2)$ to (1) we obtain an expansion of the compactly supported function $f(x)\in D(L)$ in terms of solutions to the problem of scattering theory for the operator L in the form

$$f(x)=A(x)+B(x)+\frac{2}{\pi}\int_{\Gamma}s^2ds\int_{W}\psi(x,\omega,-s)\,\Phi(\omega,s)\,d\omega.$$

$$A(x)=\sum_{j=1}^{p}\oint_{|s-k_j|=\delta}s^2ds\int_{E_3}G(x,y,s)\,f(y)\,dy,$$

$$B(x)=\sum_{j=1}^{l_2}\oint_{|s-\upsilon_j|=\delta}s^2ds\int_{E_3}G(x,y,s)\,f(y)\,dy,$$

k_j-eigenvalues, υ_j- spectral singularities of the operator L.

Let us introduce the following notation

$$J_N^f(x)=\int_0^N s^2ds\int_W\left[\psi(x,\omega,-s)\,\Phi(\omega,s)-e^{-is(x,\omega)}\hat{f}(s,\omega)\right]d\omega,$$

$$\hat{f}(s,\omega)=\int_{E_3}f(x)\,e^{i\lambda(x,\omega)}dx.$$

The following theorems have been proved:

Theorem 1. *Under the condition $C_1<4\pi$ operator $L=-\Delta+q(x)$ has neither a discrete spectrum nor spectral singularities.*

Theorem 2. *Let $f(x)\in L_1(E_3)\bigcap L_2(E_3)$. Then*

$$\lim_{N\to\infty}\left|J_N^f(x)\right|=0$$

uniformly over $x\in E_3$.

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Regularization of necessary conditions of solvability of a boundary value problem for a three-dimensional elliptic equation with variable coefficients

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Let us consider the three-dimensional elliptic equation with variable coefficients in a convex in the direction x_3 domain $D \subset R^3$ with Lyapunov boundary (surface) Γ:

$$Lu = \Delta u(x) + \sum_{k=1}^{3} a_k(x) \frac{\partial u(x)}{\partial x_k} + a(x)u(x) = 0, \ x \in D \subset R^3 \tag{1}$$

with non-local boundary conditions:

$$l_i u = \sum_{k=1}^{2} \sum_{j=1}^{3} \alpha_{ij}^{(k)}(x') \frac{\partial u(x)}{\partial x_j} \left|_{x_3=\gamma_k(x')} \right. +$$

$$+ \sum_{k=1}^{2} \alpha_i^{(k)}(x') u(x) \left|_{x_3=\gamma_k(x')} \right. = \varphi_i(x'), \ i = 1, 2; \ \ x' = (x_1, x_2) \in S, \tag{2}$$

$$u(x) = f_0(x), \ x \in \bar{\Gamma}_1 \cap \bar{\Gamma}_2 = L, \tag{3}$$

where $S = proj_{Ox_1x_2} D$, $\Gamma = \partial D = \Gamma_1 \bigcup \Gamma_2$: $\Gamma_k = \{\xi \in R^3 : \xi_3 = \gamma_k(\xi'), \xi' \in S\}$; $\gamma_k(\xi') \in C^2(S)$, $k = 1, 2$; $\alpha_{ij}^{(k)}(x') \in H^{\lambda}(S), \alpha_i^{(k)}(x'), \alpha_j(x'), a(x'), \varphi_i(x') \in C(S), i, \ k = 1, 2; \ j = \overline{1, 3}, \ f_0(x') \in C(L)$.

The fundamental solution for equation (1) is the same as for the three-dimensional Laplace equation:

$$U(x - \xi) = -\frac{1}{4\pi} \frac{1}{|x - \xi|}. \tag{4}$$

Multiplying equation (1) by the fundamental solution (4), integrating it over the domain D by parts taking into account that $\Delta_x U(x - \xi) = \delta(x - \xi)$ where $\delta(x - \xi)$ is the Dirac δ-function and taking into account $\frac{\partial U(x-\xi)}{\partial x_j} =$

$-\frac{x_j-\xi_j}{4\pi|x-\xi|^3} = -\frac{\cos(x-\xi,x_j)}{4\pi|x-\xi|^2}$ we obtain ($\xi \in \Gamma$) the 1^{st} necessary condition of solvability of problem (1)-(3):

$$u(\xi)\left|_{\xi_3=\gamma_k(\xi')}\right. = -\int_S \frac{u(x)\left|_{x_3=\gamma_k(x')}\right.}{2\pi\left|x'-\xi'\right|^2}\times$$

$$\times\frac{\cos(x-\xi,\nu_x)\Bigg|_{\substack{\xi_3=\gamma_k(\xi')\\ x_3=\gamma_k(x')}}}{P_k(x',\xi')}\frac{dx'}{\cos(\nu_x,x_3)}+...,\ k=1,2,. \tag{5}$$

Thus we have proved

Theorem 1. *Let a convex along the direction x_3 domain $D\subset R^3$ be bounded with the boundary Γ which is a Lyapunov surface. Then the obtained first necessary condition* (5)*is regular*.

Multiplying (1) by $\frac{\partial U(x-\xi)}{\partial x_i}$, $i=\overline{1,3}$,, integrating it over the domain D we obtain the rest of three necessary conditions of solvability of the problem (1)-(3):

$$\frac{\partial u}{\partial \xi_i}\left|_{\xi_3=\gamma_k(\xi')}\right. = (-1)^k\int_S \frac{\partial u(x)}{\partial x_m}\left|_{x_3=\gamma_k(x')}\right. \frac{K_{mi}(x,\xi)}{2\pi\left|x-\xi\right|^2}\Bigg|_{\substack{x_3=\gamma_k(x')\\ \xi_3=\gamma_k(\xi')}}\frac{dx'}{\cos(\nu_x,x_3)}+$$

$$+(-1)^{k+1}\int_S \frac{\partial u(x)}{\partial x_l}\left|_{x_3=\gamma_k(x')}\right. \frac{K_{li}(x,\xi)}{2\pi\left|x-\xi\right|^2}\Bigg|_{\substack{x_3=\gamma_k(x')\\ \xi_3=\gamma_k(\xi')}}\frac{dx'}{\cos(\nu_x,x_3)}+...,\ k=1,2. \tag{6}$$

where $i=\overline{1,3}$ and the numbers *i,m,l* make a permutation of numbers 1,2,3 and

$$K_{ij}(x,\xi) = \left(\cos(x-\xi,x_i)\cos(\nu_x,x_j) - \cos(x-\xi,x_j)\cos(\nu_x,x_i)\right).$$

Theorem 2. *Under assumptions of Theorem* 2.1 *the second necessary conditions* (6) of problem (1)-(3) *are singular*.

The regularization of the necessary conditions is carried out by the original scheme: we build a linear combination of necessary conditions (6) for k=1,2

(i=1,2; j=1,2,3) with unknown coefficients $\beta_{ij}^{(k)}(\xi')$ and bracket the common factor $\frac{1}{2\pi|x'-\xi'|^2}$ under the sign of integral ($i = 1,2$) and after some transformations reduce the linear combination to the form

$$\sum_{j=1}^{3}\left(\beta_{ij}^{(1)}(\xi')\frac{\partial u(\xi)}{\partial \xi_j}\left|_{\xi_3=\gamma_1(\xi')}\right.+\beta_{ij}^{(2)}(\xi')\frac{\partial u(\xi)}{\partial \xi_j}\left|_{\xi_3=\gamma_2(\xi')}\right.\right)=$$

$$=\int_S \frac{1}{2\pi\left|x'-\xi'\right|^2}\frac{dx'}{\cos(\nu_x,x_3)}\sum_{k=1}^{2}(-1)^k\times$$

$$\times\sum_{j=1}^{3}\frac{\partial u(x)}{\partial x_j}\left|_{x_3=\gamma_k(x')}\right.\left(\beta_{il}^{(k)}(x')\frac{K_{lj}^{(k)}(x',x')}{P_k(x',x')}+\beta_{im}^{(k)}(x')\frac{K_{mj}^{(k)}(x',x')}{P_k(x',x')}\right)+..., \quad (7)$$

where i=1,2 and the numbers $j,\ l,\ m$ form a permutation of numbers 1,2,3.

To regularize the integral in the right-hand side of (7) we use the boundary conditions (6) and impose the following conditions on the coefficients $\beta_{ij}^{(k)}(\xi')$:

$$(-1)^k\beta_{il}^{(k)}(x')\frac{K_{lj}^{(k)}(x',x')}{P_k(x',x')}+(-1)^k\beta_{im}^{(k)}(x')\frac{K_{mj}^{(k)}(x',x')}{P_k(x',x')}=\alpha_{ij}^{(k)}(x'), \quad (8)$$

k=1,2; j=1,2,3, where the numbers $j,\ l,\ m$ form a permutation of numbers 1,2,3 as we mentioned above. Then relationships (7) get the form:

$$\sum_{j=1}^{3}\left(\beta_{ij}^{(1)}(\xi')\frac{\partial u(\xi)}{\partial \xi_j}\left|_{\xi_3=\gamma_1(\xi')}\right.+\beta_{ij}^{(2)}(\xi')\frac{\partial u(\xi)}{\partial \xi_j}\left|_{\xi_3=\gamma_2(\xi')}\right.\right)=$$

$$=\int_S \frac{\varphi_i(x')}{2\pi\left|x'-\xi'\right|^2}\frac{dx'}{\cos(\nu_x,x_3)}-$$

$$-\int_S \frac{1}{2\pi\left|x'-\xi'\right|^2}\left[\sum_{k=1}^{2}\alpha_i^{(k)}(x')u(x',\gamma_k(x'))\right]\frac{dx'}{\cos(\nu_x,x_3)}.... \quad (9)$$

Theorem 3. *Under the conditions of Theorem* 1 *and if* 1) *boundary conditions* (2) *are linear independent*, 2) $\alpha_{ij}^{(k)}(x')\in H^{\lambda}(S)$;$\alpha_i^{(k)}(x')$,$\alpha_j(x')$, $a(x')$,$\in C(S)$, $i,\ k=1,2$; $j=\overline{1,3}$, *functions* $\varphi_i(x')$, $i=1,2$, *are continuously differentiable and vanish on the boundary* $\partial S=\bar{S}\backslash S$, 3) *system*(8) *has the unique solution* $\beta_{ij}^{(k)}(x')$, $i,k=1,2$; $j=1,2,3$, *then relationships* (9), or (7), *are regular.*

Oscillation properties of solutions of some initial-boundary value problems

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We consider the following initial-boundary value problem

$$y^{(4)}(x) - (q(x)y'(x))' = \lambda y(x),\ x \in (0,1), \tag{1}$$

$$y''(0) = y''(1) = Ty(0) - a\lambda y(0) = 0, \tag{2}$$

where $\lambda \in \mathbb{C}$ is spectral parameter, $Ty \equiv y''' - qy'$, $q(x)$ is positive absolutely continuous function on $[0,l]$, and a is a real positive constant.

We introduce the boundary condition

$$y(1)\cos\delta - Ty(1)\sin\delta = 0, \tag{3}$$

where $\delta \in [0, \pi/2]$.

By making the change of variables $t = 1 - x$, from [1, Theorem 4.1] we obtain that the eigenvalues of the problem (1)-(3) are real, simple and form an infinitely increasing sequence $\lambda_1(\delta) < \lambda_2(\delta) < \ldots$, such that $\lambda_1(\delta) > 0$ for $\delta \in [0, \pi/2)$ and $\lambda_1(\pi/2) = 0$. Moreover, the eigenfunction $y_k(x)$, corresponding to the eigenvalue $\lambda_k(\delta)$ has exactly $k-1$ simple zeros in $(0,1)$.

Lemma 1. *For each fixed $\lambda \in \mathbb{C}$ there exists a non-trivial solution $y(x,\lambda)$ of the problem (1), (2), which is unique up to a constant factor. Moreover, $y(x,\lambda)$ is an entire function of λ for each fixed $x \in [0,1]$.*

Consider the equation

$$y(x,\lambda) = 0,$$

for $x \in [0,1]$ and $\lambda \in \mathbb{R}$. It is obvious that the zeros of this equation are functions of the parameter λ.

Lemma 1. *The zeros in $(0,1]$ of the function $y(x,\lambda)$ are simple and C^1 function of $\lambda \in \mathbb{R}$.*

Lemma 2. *As $\lambda > 0$ ($\lambda \le 0$) varies the function $y(x,\lambda)$ can lose or gain zeros only by these zeros leaving or entering the interval $[0,1]$ through its endpoint $x = 1$ ($x = 0$).*

By $s(\lambda)$ we denote the number of zeros of $y(x,\lambda)$ contained in $(0,1)$.

Lemma 3. *Let* $\lambda \in (\lambda_{k-1}(0), \lambda_k(0)] \cap [0,+\infty)$, $k \in \mathbb{N}$, *then* $s(\lambda) = k-1$, *where* $\lambda_0(0) = -\infty$.

Let $\lambda < 0$ be some number, μ be a real eigenvalue of the equation (1) with the boundary conditions

$$y(0) = y''(0) = Ty(0) = y''(1) = 0, \tag{4}$$

and $\epsilon > 0$ be the sufficiently small number. The oscillation index of the eigenvalue μ is the difference between the number of zeros of the function $y(x,\lambda)$ for $\lambda \in (\mu-\epsilon,\mu)$ belonging to the interval $(0,1)$ and the number of the same zeros for $\lambda \in (\mu,\mu+\epsilon)$ [2].

Theorem 1. *There exists* $\xi < 0$ *such that the eigenvalues* $\xi_k, , k = 1, 2, \ldots,$ *of problem (1), (4) are contained in the interval* $(-\infty, \xi)$, *are simple, form an infinitely decreasing sequence and have oscillation index 1.*

Theorem 2. *Let* $i(\xi_k,)\, k \in \mathbb{N}$, *be the oscillation index of the eigenvalue* ξ_k. *Then for every* $\lambda < 0$ *the following relation holds:*

$$s(\lambda) = \sum_{\xi_s \in (\lambda, 0)} i(\xi_s).$$

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Analyzing dynamics and determining solutions for nonlinear twenty-fourth order difference equations

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This talk examines discrete-time systems, which are sometimes used to explain nonlinear natural phenomena in the sciences. Specifically, we investigate the periodicity, boundedness, oscillation, stability, and exact solutions of nonlinear difference equations. We obtain these solutions using the standard iteration method and test the stability of equilibrium points using well-known theorems. We also provide numerical examples to validate our theoretical work and implement the numerical component using Wolfram Mathematica. The method presented can be easily applied to other rational recursive problems.

In this, we explore the dynamics of adhering to rational difference formula

$$x_{n+1} = \frac{x_{n-23}}{\pm 1 \pm x_{n-5}x_{n-11}x_{n-17}x_{n-23}}, \tag{14}$$

where the initials are arbitrary nonzero real numbers.

Difference equations of type (14) have been studied by many mathematicians (see, [1, 2, 3, 5, 4, 6, 7, 8] and others).

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Parabolic problem in a non-canonical domain degenerating to a point at the initial moment of time

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We study the boundary value problem for the two-dimensional heat equation in a non-canonical domain, the boundary of which changes according to the power law $x = t^{\omega}$, $\omega > \frac{1}{2}$. There is no problem solution domain at the initial time, that is, it degenerates into a point. Using the method of generalized heat potentials, the problem is reduced to a pseudo-Volterra integral equation of the second kind. The obtained integral equation is fundamentally different from the classical Volterra integral equations in that the norm of the corresponding integral operator is equal to one and the classical method of successive approximations is not applicable to it, and the corresponding homogeneous integral equation has a non-zero solution [1-7].

We consider the following boundary value problem in the domain $Q = \left\{(r,t)|\ 0 < r < t^{\omega},\ 0 < t < \infty,\ \omega > \frac{1}{2}\right\}$:

$$\frac{\partial u}{\partial t} = a^2 \cdot \frac{1-2\beta}{r} \cdot \frac{\partial u}{\partial r} + a^2 \cdot \frac{\partial^2 u}{\partial r^2}, \tag{1}$$

$$u(r,t)|_{r=0} = g_1(t), \quad t > 0, \tag{2}$$

$$u(r,t)|_{r=t^{\omega}} = g_2(t), \quad t > 0, \tag{3}$$

where $0 < \beta < 1$.

Theorem 1. *Let the conditions $g_1(t) \in M(0,+\infty)$, $t^{\omega(1-\beta)} g_2(t) \in M(0,+\infty)$, $M(0,+\infty) = L_{\infty}(0,+\infty) \cap C(0,+\infty)$ are satisfied, then the boundary value problem (1)-(3) has a solution*

$$u(r,t) = \int_0^t \frac{\partial G(r,\xi,t-\tau)}{\partial \xi}\bigg|_{\xi=\tau^{\omega}} \mu(\tau)d\tau + \int_0^t \frac{\partial G(r,\xi,t-\tau)}{\partial \xi}\bigg|_{\xi=0} \nu(\tau)d\tau,$$

where

$$G(r,\xi,t-\tau) = \frac{1}{2a^2} \cdot \frac{r^{\beta} \cdot \xi^{1-\beta}}{t-\tau} \cdot \exp\left[-\frac{r^2+\xi^2}{4a^2(t-\tau)}\right] \cdot I_{\beta}\left(\frac{r\xi}{2a^2(t-\tau)}\right),$$

$$\nu(t) = 2a^2 \beta g_1(t),$$

and $\mu(t)$ is determined from the following pseudo-Volterra integral equation:

$$\mu(t) - \int_0^t N(t,\tau)\mu(\tau)d\tau = f(t),$$

where

$$N(t,\tau) = \frac{t^{\omega\beta}\tau^{\omega(1-\beta)}(t^{\omega}-\tau^{\omega})}{2a^2(t-\tau)^2} \exp\left[-\frac{(t^{\omega}-\tau^{\omega})^2}{4a^2(t-\tau)}\right] \exp\left[-\frac{t^{\omega}\tau^{\omega}}{2a^2(t-\tau)}\right] \cdot I_{\beta}\left(\frac{t^{\omega}\tau^{\omega}}{2a^2(t-\tau)}\right) +$$

$$+\frac{t^{\omega(\beta+1)}\tau^{\omega(1-\beta)}}{2a^2(t-\tau)^2} \exp\left[-\frac{(t^{\omega}-\tau^{\omega})^2}{4a^2(t-\tau)}\right] \exp\left[-\frac{t^{\omega}\tau^{\omega}}{2a^2(t-\tau)}\right] \cdot I_{\beta-1,\beta}\left(\frac{t^{\omega}\tau^{\omega}}{2a^2(t-\tau)}\right) +$$

$$+\frac{t^{\omega\beta}(1-2\beta)}{(t-\tau)\tau^{\omega\beta}}\exp\left[-\frac{(t^{\omega}-\tau^{\omega})^2}{4a^2(t-\tau)}\right]\exp\left[-\frac{t^{\omega}\tau^{\omega}}{2a^2(t-\tau)}\right]\cdot I_{\beta}\left(\frac{t^{\omega}\tau^{\omega}}{2a^2(t-\tau)}\right),$$

$$I_{\beta-1,\beta}(z)=I_{\beta-1}(z)-I_{\beta}(z),$$

$$f(t)=-2a^2g_2(t)+2a^2\widetilde{g_1}(t^{\omega},t),$$

$$\widetilde{g_1}(r,t)=\frac{1}{(2a^2)^{\beta}}\cdot\frac{1}{2^{\beta}}\cdot\frac{1}{\Gamma(\beta)}\int_0^t\frac{r^{2\beta}}{(t-\tau)^{\beta+1}}\cdot\exp\left[-\frac{r^2}{4a^2(t-\tau)}\right]\cdot g_1(t)d\tau,$$

where $I_{\nu}(z)$ *– is a modified Bessel function of order* ν*.*

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Nodal solutions of some nonlinear Dirac systems

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Consider the following nonlinear Dirac system

$$Bw'(x) - P(x)w(x) = sf(w(x)),\ 0 < x < \pi, \tag{1}$$

$$U_1(w) := (\sin\alpha,\ \cos\alpha)\, w(0) = v(0)\cos\alpha + u\,(0)\sin\alpha\ = 0, \tag{2}$$

$$U_2(w) := (\sin\beta,\ \cos\beta) w(\pi) = v(\pi)\cos\beta + u\,(\pi)\sin\beta = 0, \tag{3}$$

where

$$B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ P(x) = \begin{pmatrix} p\,(x) & 0 \\ 0 & r(x) \end{pmatrix},\ w(x) = \begin{pmatrix} u(x) \\ v(x) \end{pmatrix},$$

$p\,(x),\ r(x) \in C([0,\pi];\mathbb{R})$, $\alpha,\ \beta$, are real constants such that $0 \le \alpha,\ \beta < \pi$. Here s is a nonzero real number and the function $f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in C\,(\mathbb{R}^2 \times \mathbb{R}^2; \mathbb{R}^2)$ satisfies the following condition:

$$f(w) = \gamma w + h(w) \text{ and } f(w) = \delta w + \hbar(w), \tag{4}$$

where $\gamma,\ \delta\ (\gamma \ne \delta)$ are positive constants and

$$h(w) = \begin{pmatrix} h_1(w) \\ h_2(w) \end{pmatrix} = o(|w|) \text{ as } |w| \to 0, \tag{5}$$

$$\hbar(w) = \begin{pmatrix} \hbar_1(w) \\ \hbar_2(w) \end{pmatrix} = o(|w|) \text{ as } |w| \to \infty \tag{6}$$

(here by $|\cdot|$ we denote the norm in $\mathbb{R}^2$).

The Dirac equations describing spin-1/2 particles, such as electrons and positrons, play an important role in both physics and mathematics. The nonlinear Dirac equations describe the self-action in quantum electrodynamics and are used as an effective theory in atomic and gravitational physics (see [1]).

Let E be the Banach space $C\left([0,\pi];\,\mathbb{R}^2\right) \cap \left\{w\,:\,U(w) = \tilde{0}\right\}$ with the norm $||w||_0 = \max\limits_{x\in[0,\,\pi]} |u(x)| + \max\limits_{x\in[0,\,\pi]} |v(x)|$, where $U(w) = \begin{pmatrix} U_1(w) \\ U_2(w) \end{pmatrix}$, $\tilde{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

We denote by S a subset of E, which defined by

$$S = \left\{w \in E:\ |u(x) + |v(x)|\ > 0,\ \forall x \in [0,\pi]\right\}.$$

As in [2, p. 313], for each $w = \begin{pmatrix} u \\ v \end{pmatrix} \in S$ we define a continuous function $\theta(w,x)$ on $[0,\pi]$ by

$$\tan\theta(w,x) = \frac{v(x)}{u(x)},\ \theta(w,0) = -\,\alpha. \tag{7}$$

Along with problem (1)-(3), we consider the following linear Dirac system

$$\begin{cases} Bw'(x) - P(x)w(x) = \lambda w(x),\ x \in (0,\pi), \\ U(w) = \tilde{0}, \end{cases} \tag{8}$$

where $\lambda \in \mathbb{C}$ is an eigenvalue parameter.

Remark 1. It follows from [2, Theorem 3.1] that the eigenvalues $\lambda_k, k\in\mathbb{Z}$, of the problem (8) are real and simple, and can be numbered in ascending order on the real axis

$$\ldots\ < \lambda_{-k} <\ \ldots\ < \lambda_{-1} < \lambda_0 < \lambda_1 <\ \ldots\ < \lambda_k <\ \ldots\ .$$

Moreover, for the eigenvector-function $w_k(x)$, $k \in \mathbb{Z}$, corresponding to the eigenvalue λ_k, the angular function $\theta(w_k, x)$ at $x = \pi$ satisfy the condition

$$\theta(w_k,\pi) = -\,\beta + k\pi. \tag{8}$$

For each $k \in \mathbb{Z}$ and each ν, let S_k^ν denote the set of vector-functions $w \in S$ with the following properties:

(i) $\theta(w, \pi) = -\beta + k\pi$;

(ii) if $k > 0$ or $k = 0, \alpha \geq \beta$ (except the cases $\alpha = \beta = 0$ and $\alpha = \beta = \pi/2$), then for fixed w, as x increases from 0 to π, the function θ cannot tend to a multiple of $\pi/2$ from above, and as x decreases, the function θ cannot tend to a multiple of $\pi/2$ from below; if $k < 0$ or $k = 0, \alpha < \beta$, then for fixed w, as x increases, the function θ cannot tend to a multiple of $\pi/2$ from below, and as x decreases, the function θ cannot tend to a multiple of $\pi/2$ from above;

(iii) the function $\nu u(x)$ is positive in a deleted neighborhood of $x = 0$.

Remark 2. By (7), Remark 1 (see (9)) and [2, Theorem 2.1 and the proof of the second part of Theorem 3.1] we have $w_k \in S_k = S_k^- \cup S_k^+$, i.e. the sets S_k^-, S_k^+ and S_k are nonempty. From the definition of the sets S_k^-, S_k^+ and S_k it can be seen that they are disjoint and open in E.

In this note, for each $k \in \mathbb{Z}$ and each ν, an interval is determined for s in which there are solutions to the problem (1)-(3) contained in the set S_k^ν.

Theorem 1. *Let conditions (4)-(6) hold,* $\lambda_k \neq 0$ *and*

$$\frac{\lambda_k}{\gamma} < s < \frac{\lambda_k}{\delta} \text{ or } \frac{\lambda_k}{\delta} < s < \frac{\lambda_k}{\gamma}$$

for some $k \in \mathbb{Z}$. *Then problem has two solutions* w_k^+ *and* w_k^- *such that* $w_k^+ \in S_k^+$ *and* $w_k^- \in S_k^-$.

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A boundary value problem in an infinite strip for a biharmonic equation degenerating into a single characteristic equation

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Mutual degenerations of single characteristic and elliptic equations in bounded domains have been studied by M.I. Vishik and L.A. Lusternik in [1]. In the present paper in an infinite strip $\Pi = \{(x,y)|\ 0 \le x \le 1,\ -\infty < y < +\infty\}$ we consider the following boundary value problem

$$\varepsilon\Delta^2 u + \frac{\partial}{\partial x}(\Delta u) + au = f(x,y), \tag{1}$$

$$u|_{x=0} = u|_{x=1} = 0,\ \left.\frac{\partial u}{\partial x}\right|_{x=0} = \left.\frac{\partial u}{\partial x}\right|_{x=1} = 0, \quad (-\infty < y < +\infty), \tag{2}$$

$$\lim_{|y|\to+\infty} u = 0, \lim_{|y|\to+\infty} \frac{\partial u}{\partial y} = 0, \quad (0 \le x \le 1), \tag{3}$$

where $\varepsilon > 0$ is a small parameter, $a > 0$ is a constant, $f(x,y)$ is a prescribed function.

We prove the following main theorem.

Theorem. *Let $f(x,y)$ be a prescribed function in Π with discontinuous derivatives with respect to x up to the $(n+4)$ the order inclusively, be infinitely differentiable with respect to the variable y and satisfy the condition:*

$$\sup_y \left(1 + |y|^l\right) \left|\frac{\partial^k f(x,y)}{\partial x^{k_1}\partial y^{k_2}}\right| = C_{lk_1k_2} < +\infty,$$

where $k = k_1 + k_2, k_1 \le n+4, C_{lk_1k_2} > 0$. Then for solving the problem (1)-(2) we have the asymptotic representation

$$u = \sum_{i=0}^{n} \varepsilon^i W_i + \sum_{j=0}^{n+3} \varepsilon^j V_j + \varepsilon^{n+1} z.$$

Here the functions W_i are determined by the first iterative process as the solutions of the problems

$$\frac{\partial}{\partial x}(\Delta W_i) + aW_i = f_i(x,y),$$

$$W_i|_{x=0} = W_i|_{x=1} = 0, \quad \left.\frac{\partial W_i}{\partial x}\right|_{x=1} = 0,$$

$$\lim_{|y|\to+\infty} W_i = 0,$$

where $i=0$, $f_0(x,y) = f(x,y)$, for $i = 1,2,...,n$ the functions $f_i(x,y)$ depend on $W_0, W_1, ..., W_{i-1}, V_j$ are boundary layer functions near the boundary $x=0$, that are determined by the second iterative process, $\varepsilon^{n+1}z$ is a residual term, and for z the following estimation is valid,

$$\varepsilon\iint\limits_{\Pi}\left[\left(\frac{\partial^2 z}{\partial x^2}\right)^2 + \left(\frac{\partial^2 z}{\partial y^2}\right)^2\right]dxdy + \int\limits_{-\infty}^{+\infty}\left(\left.\frac{\partial z_i}{\partial x}\right|_{x=0}\right)dy + a\iint\limits_{\Pi} z^2 dxdy \le C\varepsilon^{n+1},$$

where $C>0$ is a constant independent of ε.

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A boundary value problem for higher order elliptic equation degenerating into a lower order elliptic equation

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Let $(n-1)$ dimensional surface S of class C^∞ divide the bounded domain $\Omega \subset R^n$ with a rather smooth boundary Γ into the domains Ω_1 and Ω_2 . In Ω Γ we consider the following boundary value problem

$$L_\varepsilon u \equiv \varepsilon^{2(l-k)} L_{2l} u + L_{2k} u = f(x) \tag{1}$$

$$\left.\frac{\partial^i u}{\partial \nu^i}\right|_\Gamma = 0; i = 0, 1, ..., 2l-1, \tag{2}$$

where $\varepsilon > 0$ is a small parameter,

$$\begin{aligned} L_{2k} &= \sum_{|\alpha| \le 2k} a_\alpha(x) D^\alpha, \quad L_{2l} = \sum_{|\alpha| \le 2k} b_\alpha(x) D^\alpha, \ \alpha = (\alpha_1, \alpha_2, ..., \alpha_n), \\ |\alpha| &= \sum_{j=1}^n \alpha_j, \ D^\alpha = D_1^{\alpha_1}, D_2^{\alpha_2}, ..., D_n^{\alpha_n}, D_j = \frac{\partial}{\partial x_j}, \quad x = (x_1, x_2, ..., x_n), \ l > k, \end{aligned}$$

ν is a normal to Γ, $f(x)$ is a prescribed rather smooth function for $x \in \Omega_p, (p = 1, 2)$, possibly having discontinuities of first kind on S .

It is assumed that the following conditions are fulfilled:

1) The coefficients a $a_\alpha(x)$, $b_\alpha(x)$ are rather smooth and the polynomials

$$P_{2l} = \sum_{|\alpha| \le 2k} a_\alpha(x) \xi^\alpha, \quad Q_{2k} = \sum_{|\alpha| \le 2k} b_\alpha(x) \xi^\alpha$$

are non-zero for $|\xi| \neq 0$ and have the same signs in $\overline{\Omega}$, where $\xi = (\xi_1, \xi_2, ..., \xi_n)$ ξ_i are real numbers, $\xi^\alpha = \xi_1^{\alpha_1}, \xi_2^{\alpha_2}, ..., \xi_n^{\alpha_n}$.

2) $(L_{2l} u, u) \ge \alpha_1^2 (u, u)$, where α_1^2 is independent of u, while u satisfied the conditions (2).

3) $(L_{2k}, w, w) \geq \alpha_2^2 \left[\sum\limits_{j=1}^{k=1} (D^j w, D^j w) + (w, w)\right]$, where α_2^2 is independent of w while w satisfies the first of conditions from (2).

Construction of the asymptotics of the first boundary value problem for elliptic equations degenerating into a lower order elliptic equation in the absence of a special surface S, was stated in detail in [1].

Conditions 1)-3) yield that problem (1),(2) has a unique solution. Following the works [2],[3] we can show that the solution of problem (1),(2) has continuous derivatives to the $(2l-1)$-th order inclusively: $u(x) \in C^{2l-1}(\overline{\Omega})$.

The degenerated problem corresponding to problems (1), (2) is of the form

$$L_{2k} w = f, \tag{3}$$

$$\left.\frac{\partial^i w}{\partial \nu^j}\right|_{\Gamma} = 0;\ j = 0, 1, ..., k-1, \tag{4}$$

Obviously, $w \in C^{2k-1}(\overline{\Omega})$. The functions $D^{2k}w, D^{2k+1}w, ..., D^{2l-1}w$ have discontinuities on the surface S, i.e. for $2k, 2k+1, ..., (2l-1)$ -th derivatives of the solutions $u(x)$ the phenomenon of the internal boundary layer is observed. To compensate for these discontinuities, a boundary layer type function η should be constructed near the surface S.

In addition, since $\frac{\partial^k w}{\partial \nu^k}, \frac{\partial^{k+1} w}{\partial \nu^{k+}}, ..., \frac{\partial^{2l-1} w}{\partial \nu^{2l-1}}$ does not satisfy, generally speaking, the boundary conditions on Γ, then in the vicinity of Γ for the function $u(x)$ we can still, observe the phenomenon of external boundary layer. Therefore we should construct one more boundary layer type function V near the boundary Γ.

We proved the following theorem.

Theorem. Let $(n-1)-$ dimensional surface S of class C^{∞} divide the bounded domain $\Omega \subset R^n$ with a rather smooth boundary Γ into two parts , and $f(x_1, x_2, ..., x_n)$ be a function smooth on $\Omega \setminus S$ with first kind discontinuity on the surface S. Assume that L_{2l} and L_{2k} are elliptic differential operators with variable coefficients of orders $2l$ and $2k$, $l-k = 2p > 0$, respectively. Then, subject to conditions 1)-3) the solution of problems (1), (2) is representable in the form

$$u = w + \eta + V + \varepsilon z,$$

where w is the solution of the degenerated problem, $\eta = \varepsilon^{2k}\eta_0$ is a boundary layer function near the surface $S, V = \sum\limits_{i=1}^{m} \varepsilon^{k+i} V_i,\ (m = \max\{k, l-k-1\})$ is a boundary layer function near the boundary Γ, while εz is a residual term, and for z the following estimation is valid:

$$\|z\|_{w_2^{k-1}(\Omega)} \leq C,$$

where $c > 0$ is a constant independent of ε.

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Nonlocal initial-boundary hyperbolic problems. boundary conditions that do not allow direct use of the Fourier method

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Initial boundary value problems for linear equations of hyperbolic type are a fairly well-developed part of the theory of partial differential equations. And one of more developed methods of their solution is the Fourier method which is also called the method of separation of variables or the eigenfunction expansion method. This method has been well developed for the case of self-adjoint boundary conditions with respect to the spatial variable.

The task is to study the well-posedness of formulation of boundary value problems for (one-dimensional) partial differential equations of hyperbolic type with nonlocal boundary conditions that do not allow the direct use of the Fourier method (method of separation of variables).

The author's research concept is that it is necessary to construct a method for solving nonlocal boundary value problems for a wave equation for the case when the system of root functions of the corresponding spectral problem (that arise when using the method of separation of variables) does not form an unconditional basis. In this regard, the direct use of the Fourier method turns out to be impossible.

The main hypothesis is that any problem for a wave equation with not strongly regular boundary conditions (with respect to the spatial variable) can be reduced to a sequential solution of two problems with strongly regular boundary conditions. The hypothesis is based on the previously obtained (by the author) results for the case of a one-dimensional heat conduction equation [1].

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Nonexistence of global solutions of Cauchy problems for systems of semilinear Klein–Gordon equations

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Among the nonlinear hyperbolic equations, the Klein-Gordon equation has an important theoretical and practical meaning. The nonlinear Klein-Gordon equation appears in the study of a number of problems of mathematical physics. For example, this equation appears in general relativity, nonlinear optics, plasma physics, fluid mechanics, radiation theory, and other issues.

The existence and nonexistence of global solutions for the Cauchy problem for the wave equation

$$u_{tt} - \Delta u + mu + u_t = f(u), \quad t > 0, \ x \in R^n, \tag{1}$$

with initial condition

$$u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \quad x \in R^n, \tag{2}$$

and a damping term, we refer the reader to [1, 2, 3, 4].

The absence of global solutions with positive arbitrary initial energy for systems of semilinear hyperbolic equations

$$u_{itt} - \Delta u_i + u_i + \gamma u_{it} = \sum_{\substack{i,j=1 \\ i \neq j}}^{m} |u_j|^{p_j} |u_i|^{p_i} u_i, \quad i = 1, 2, ..., m$$

was investigated in [6], where $n \geq 2, \ p_j \geq 0, j = 1, 2, ..., m$ and $\sum\limits_{k=1}^{m} p_k \leq \frac{2}{n-2}$ if $n \geq 3$.

In the domain $[0, \infty) \times R^n$ we consider the Cauchy problem for a system of semilinear Klein-Gordon equations

$$u_{itt} - \Delta u_i + q_i u_i + \gamma_i u_{it} = \lambda_i \prod_{\substack{j=1 \\ j \neq i}}^{m} |u_j|^{p_j+1} |u_i|^{p_j-1} u_i, \ i = 1, ..., m \tag{3}$$

with initial condition

$$u_i(0,x) = u_{i0}(ix), \quad u_{it}(0,x) = u_{i1}(ix), \quad x \in R^n, \quad i = 1, ..., m. \tag{4}$$

Where $u_1, ..., u_m$ are real functions depending on $t \in R_+, x \in R^n$ are real numbers.

Assume that

$$p_j > 0, \quad j = 1, ..., m, \ m = 2, 3, ..., \tag{5}$$

$$\sum_{k=1}^{m} p_k + m - 2 \leq \frac{2}{n-2} \text{ if } n \geq 3. \tag{6}$$

We study qualitative characteristics of the family of the potential wells, the existence and nonexistence of global solutions of problem (3),(4). Similar problems for a systems of Klein-Gordon with two equations have been studied in [5], and for a systems of Klein-Gordon with m equations were studied in [7].

In the sequel, by $|\,.\,|_q$ we denote the usual $L_q(R^n)$-norm. For simplicity of notation, in particular, we write $|\,.\,|_q$ instead of $|\,.\,|$. The scalar product in $L_2(R^n)$ will be denoted by $\langle .,. \rangle$. After we will denote the norm in the Sobolev space $H^1 = W_2^1(R^n)$, i.e. by $\|u\| = \left[\left| |\nabla u|^2 + |u|^2 \right| \right]^{1/2}$, where ∇ is the gradient.

For simplicity let $q_j = \gamma_i = \lambda_i = 1,\ j = 1, ..., m$.

Let $(\bar{\varphi}_1, ..., \bar{\varphi}_m)$ be the solution of the system (1). Then it is clear that $(u_1(t,x), ..., u_m(t,x)) = (\bar{\varphi}_1, ..., \bar{\varphi}_m)$ the solution of system (3) satisfying the initial conditions $u_1(0,x) = \bar{\varphi}_1(x), ..., u_m(0,x) = \bar{\varphi}_m(x),\ x \in R^n$.

In this case $(\bar{\varphi}_1, ..., \bar{\varphi}_m)$ called a stagnant solution to the problem (3),(4).

We define the following functionals

$$J(\varphi_1, ..., \varphi_m) = \sum_{j=1}^{m} \frac{p_j + 1}{2} \|\varphi_j\|^2 - G,\ I(\varphi_1, ..., \varphi_m) = \sum_{j=1}^{m} \frac{p_j + 1}{\sum\limits_{\mu=1}^{m} p_\mu + m} \|\varphi_j\|^2 - G.$$

Here $G = G(\varphi_1, ..., \varphi_m) = \int_{R^n} \prod\limits_{j=1}^{m} |\varphi_j(x)|^{p_j+1}\, dx$.

We denote by $E(t)$ the following energy functional
$E(t) = \sum\limits_{j=1}^{m} \frac{p_j+1}{2} \left[\left| u'_{jt}(t,\cdot) \right|^2 + \|u_j(t,\cdot)\|^2 \right] - \int_{R^n} \prod\limits_{j=1}^{m} |u_j(t,x)|^{p_j+1}\, dx.$

The main result of the present paper is stated in the following statement.

Theorem. *Let conditions (5),(6) be satisfied and*

$$E\,(0) > 0, I_\delta(u_{10}, ..., u_{m0}) < 0,\ \sum_{j=1}^{m} \frac{p_j + 1}{2} \|u_{j0}\|^2 > \frac{\sum\limits_{\mu=1}^{m} p_\mu + m - 2}{p_\mu + m} E(0).$$

Then the solution of the Cauchy problem (3), (4) blows up in finite time.

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Existence and stability of solution for time-delayed nonlinear fractional differential equations

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The objective of this study is to analyze the existence and stability of solutions for a mathematical model of nonlinear systems of time-delayed fractional differential equations involving Atangana-Baleanu-Caputo ($\mathcal{ABC}$) fractional derivatives with vaccination. The epidemiological states of the total population are classified as susceptible individuals $\mathcal{S}(t)$, infected individuals $\mathcal{I}(t)$, quarantined individuals $\mathcal{Q}(t)$, recovered individuals $\mathcal{R}(t)$, vaccinated individuals $\mathcal{V}(t)$ and insusceptible/protected persons $\mathcal{P}(t)$. By considering coordinate transformation, we linearize our system and obtain a matrix coefficient for the proposed time-delayed fractional differential equation. We establish sufficient requirements and inequalities using the generalized Gronwall inequality and Krasnoselskii's fixed-point theorem to prove the existence of solution for our problem. Additionally, we construct sufficient requirements for stability analysis by employing the Laplace transform and matrix measure theory. We also investigate criteria for the local stability of the epidemic and disease-free equilibrium points. Finally, we provide a numerical illustration to validate the effectiveness of our results.

On an application of Wiman-Valiron type estimate

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In the paper, we give an application of the main result obtained for evolution type parabolic equations to Cauchy-Dirichlet type boundary value problem.

In a Hilbert space, we consider evolution types such differential-operator equations:

$$u'(t) \pm A(t)u(t) = 0, \tag{1}$$

where $A(T)$ linear positive, self-adjoint operator with a and discrete spectrum.

We introduce the following denotation:

$$M(t) = \|u(t)\|, \quad \mu(t) = \max_k |(u(t),\ \varphi_k(t))|, \tag{2}$$

where $\{\varphi_k(t)\}$ is a system of eigen-function of $A(t)$. In the Wiman-Valiron type estimate problem, we can find such class increasing positive functions $\psi(t)$ that the inequalities such as

$$M(t) \leq \psi(\mu(t))$$

be fulfilled (in definite sense).

These kinds of results have created fruitful new research in the quality theory of differential equations. In our studies, new type estimates were identified for solving parabolic, inverse-parabolic Cauchy problems and also for power series.

Assume that $f(z) = \sum\limits_{0}^{\infty} a_n z^n$ are complete functions and

$$M(r) = \max|f(z)|\ ,\ \mu(r) = \max_n |a_n|\, r^n, \quad |z| \leq r.$$

It is known that $\mu(r) \leq M(r)$. But a very important problem is the problem of estimating $M(r)$ from above by means of $\mu(r)$.

In 1914-1916, for the first time, Wiman and Valiron obtained the following classic result

$$M(r) \le \mu(r)\left[\log \mu(r)\right]^{1/2+\varepsilon}, \quad \varepsilon > 0. \tag{3}$$

So, denoting by $E \subset (0, \infty)$ such a set of numbers $r > 0$ that these inequality (4) is violated, then $\int\limits_E \frac{dr}{r} < \infty$ E is called an exceptional set. In 1963 Rosenbloom . strengthened this result. For a class od definite $\psi(y)$ functions he proved such an inequality that (out of E):

$$\frac{M(r)}{\sqrt{\psi\left(\log \mu(r)\right)}} \le \mu(r). \tag{4}$$

Just this result led us to the results obtained on Wiman-Valiron type estimations for differential equations.

The main results obtained by us for solving parabolic equations is as follows:

$$\frac{M(t)}{\sqrt{\psi\left(\log \mu(t)\right)}} \le t^{-\gamma}\mu(t),\ \gamma > 0. \tag{5}$$

where the number γ is determined by means of the operator $A(t)$. In a special case, from (5) we obtain the following result:

$$M(t) \le t^{-\gamma}\mu(t)\left[\log t^{-\gamma}\mu(t)\right]^{s}, \quad \gamma, s > 0. \tag{6}$$

So, the relations (5) and (6) can be violated only in the definite set $E \subset (0, 1)$, and $\int\limits_E \frac{dt}{t} < \infty$. Now let as consider an example of one application of the main result of (5) and (6) to the solution of a specific mathematical physics problem.

Assume that $\Omega \subset R^n$ is a domain, $Q = \Omega \times (0, T)$ is a cylinder, $\Gamma = \partial\Omega \times (0, T)$ is a lateral surface. Let us consider the following Cauchy-Dirichlet (Neumann) type boundary value problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} + Au = 0 \;\; in\ Q \\ \quad u = 0\ on\ \ \Gamma \\ \; u = u(x,0)\ in\ \ \Omega \end{cases} \tag{7}$$

$$Au = -\sum \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right) + cU,$$

$$\sum_{i,j=1}^{n} a_{ij}(x,t)\xi_i\xi_j \geq \alpha\,|\xi|^2\,, \quad \alpha > 0, \;\; \xi \in R^n, \;\; c \geq 0, \; x \in \Omega,$$

(in special case $A = -\Delta$ or $A = (-\Delta)^{\sigma/2}$, $\delta > 0$. According to the formula of exact spectral asymptotics obtained by Seeley [4] (and Hormander)

$$N(\lambda) = \lambda^{n/2} + O\left(\lambda^{\frac{n-1}{2}}\right), \;\; \lambda \to \infty, \;\; n \geq 2.$$

Hence we obtain $\gamma = \alpha + \dfrac{2s+1}{4}$, $\;s = \dfrac{n}{2} - 1$.

For example. accepting $c_a(0) = (u_0, \varphi_n) = \sqrt{n^k}$, $\lambda_n = \dfrac{n^\sigma}{2}$ In the system from the formula $u(x,t) = \sum\limits_{n=0}^{\infty} c_n(0) e^{-t\lambda_n}\varphi_n(x)$

we obtain

$$\|u\,(\cdot,t)\|^2 = \sum_n n^k e^{-t} n^\sigma = \sum_p n^p e^{-tp} = (-1)^p \frac{d^p}{dt^p}\left(\frac{1}{1-e^{-t}}\right) \tilde{} \frac{c}{t^{k+1}}, \quad t \to 0$$

$$\mu^2(t) = \max_n n^k e^{-t} n^\sigma = \frac{c}{t^k}, \quad t \to 0.$$

So, we obtain $M(t) = \frac{\mu(t)}{\sqrt{t}}, \quad t \to 0.$

The formulas of the last and most exact spectral asymptotics for the Laplace-Dirichlet (Neumann) operator in the domain $\Omega \subset R^n$ were obtained in 2005 by Ya. Safarov and Ya. Netrusov. These formulas are as follows; $N(\lambda) = \lambda^n + O\left(\lambda^{n-\alpha}\right),\ 0 < \alpha < 1$ (for Laplace -Dirichlet) $N(\lambda) = \lambda^n + O\left(\lambda^{\frac{n-1}{\alpha}}\right),\ 0 < \alpha < 1$ (for Laplace-Neumann).

Taking these into account, for the Laplace-Diricjles type boundary value problem (7) we obtain:

$$M(t) = t^{-\gamma}\mu(t)\left[\log t^{-\gamma}\mu(t)\right]^{\frac{n-1}{4}}, \;\; \gamma = \frac{1}{2} + \frac{n-1}{4}.$$

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Inverse source problem for the diffusion equation on a metric star graph

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We consider a graph $\Gamma = V \cup E$ consisting of a finite set of vertices $V = \{\nu_k\}_0^n$ and a set of edges $E = \{e_k\}_1^n$ connecting these vertices. The graph under consideration in this study is a metric star graph. Thus, each edge $e_k, k = \overline{1,n}$ is parametrised by an interval $(0, l_k)$. The coordinates x_k are defined on each bond. For the function, $u : \Gamma \to R$, defined on the graph, we put $u|_{e_k} = u_k$. We will use x instead of x_k.

We are interested in the following the time-fractional problem with the Caputo-derivative of order $\alpha \in (0,1)$ on each edge of the graph Γ

$$\partial_{0,t}^{\alpha} u_k(x,t) - u_{k,xx}(x,t) = f_k(x) g_k(x,t), \quad x \in e_k, \ t \in (0,T], \ k = \overline{1,n},$$

with initial conditions

$$u_k(x,0) = 0, \quad x \in \bar{e}_k, \quad k = \overline{1,n},$$

the vertex conditions

$$\sum_{k=1}^{n} u_{k,x}(0,t) = 0, \quad u_k(0,t) = u_j(0,t), \ k \neq j, \ k,j = \overline{1,n}, \ t \in (0,T],$$

and the boundary conditions

$$u_k(l_k,t) = 0, \ t \in (0,T], \ k = \overline{1,n}.$$

where $g_k(x,t)$, $i = \overline{1,n}$, are given functions, $f_k(x)$ are unknown functions.

The main aim is to find the pair of functions $\{u_k(x,t), f_k(x)\}, k = \overline{1,n}$. To find $f_k(x)$, which is the solution to the inverse problem, we need additional conditions. That's why, we introduce additional integral overdetermination conditions of the form

$$\int_0^T \partial_{0,t}^{\alpha} u_k(x,t) \eta(t) dt = \psi_k(x), \ k = \overline{1,n},$$

where $\eta(t)$ and $\psi_k(x),\ k=\overline{1,n}$ are given functions.

M. Mehandiratta et al. [1], studied the time- fractional diffusion equation on a metric star graph and proved some priori estimates. Inverse problem for parabolic equation satisfying an integral overdetermination condition were studied by Kamynin in [2]. In this present study, the primary focus lies on the inverse problem on a metric star graph. The approach to addressing this problem involves utilizing the method introduced by Ladyzhenskaya [3], which involves reducing the given problem to an operator equation and employing priory estimates.

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Direct and inverse problems for parabolic equations in degenerate domains

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Let's consider the following nonlinearly degenerating domain

$$\Omega=\{x,t\,|\,\varphi_1(t)<x<\varphi_2(t),\ 0<t<T<\infty\},$$

with its cross-section $\Omega_t = \{\varphi_1(t) < x < \varphi_2(t)\}$ for a fixed value of the time variable $t \in (0,T)$, with

$$\varphi_1(0) = \varphi_2(0).$$

The functions $\varphi_1(t)$ and $\varphi_2(t)$ are defined on $[0,T]$, and belong to $C^1(0,T)$.

In the domain Ω consider the following inverse problem for the Burgers equation

$$\partial_t u(x,t) + u(x,t)\partial_x u(x,t) - \nu\partial_x^2 u(x,t) = w(t)f_t(x), \text{ in } \Omega, \tag{1}$$

$$\partial_x^j u(\varphi_1(t),t) = \partial_x^j u(\varphi_2(t),t), \ j = 0,\, 1;\, t \in (0,T), \tag{2}$$

$$u(0,0) = 0, \tag{3}$$

$$\int_{\varphi_1(t)}^{\varphi_2(t)} u(x,t)dx = E(t), \ t \in (0,T), \tag{4}$$

here $\nu = const > 0$ is a given constant, the functions $f_t(x)$, $E(t)$ satisfy the next conditions

$$\begin{cases} f_t(x) \equiv f(x,t) \in L^\infty(0,T;L^\infty(\Omega_t)),\, E(t) \in W^{1,\infty}(0,T),\, E(0) = 0, \\ \widetilde{f}(t) \equiv \int\limits_{\varphi_1(t)}^{\varphi_2(t)} f_t(x)dx \neq 0,\, \forall t \in (t_0,T),\, \forall t_0 :\ 0 < t_0 < T, \end{cases}$$

where $\varphi(t) = \varphi_2(t) - \varphi_1(t)$.

We will examine direct and inverse problems for parabolic equations in degenerate domains and find conditions under which these problems will be uniquely solvable.

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Obtaining necessary conditions for the non-local boundary condition problem for a Laplace equation

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Let us consider the following boundary value problem:

$$\Delta u(x) \equiv \sum_{j=1}^{2} \frac{\partial^2 u(x)}{\partial x_j^2} = 0, \ x = (x_1, x_2) \in D \subset R^2, \tag{1}$$

$$\sum_{k=1}^{2} \left[\sum_{j=1}^{2} \alpha_{ij}^{(k)}(x_1) \frac{\partial u(x)}{\partial x_j} + \alpha_i^{(k)}(x_1) u(x) \right] \Bigg|_{x_2=\gamma_k(x_1)} =$$

$$= \alpha_i(x_1), \ i = 1, 2; \ x_1 \in [a_1, b_1], \tag{1}$$

where $i=\sqrt{-1}$, D is a bounded convex domain, Γ – boundary of the domain is a Lyapunov line, when projecting the domain D on the axis x_1 parallel to x_2, the boundary Γ is divided into two parts, Γ_1 and Γ_2. The equations of these lines $x_2=?_k(x_1)$, $k=1,2;;$ $x_1 \in [a_1, b_1]$. $\gamma_k(x_1)$- are functions with real value, derivatives of which (first order) are from the Hölder class. It is known that the fundamental solution of equation (1) is of the form [1]:

$$U(x-\xi) = -\frac{1}{2\pi} \ln|x-\xi|, \tag{3}$$

Basic relations: Multiplying the equation (1) by the fundamental solution (3), integrating with respect to the domain D, we have [2]:

$$\int_D \frac{\partial^2 u(x)}{\partial x_1^2} U(x-\xi) dx + \int_D \frac{\partial^2 u(x)}{\partial x_2^2} U(x-\xi) dx = 0$$

Applying the Ostrogradsky-Gauss formula, i.e. integration by parts, we have:

$$\int_\Gamma u(x)\frac{\partial U(x-\xi)}{\partial x_1}\cos(\nu,x_1)\,dx-\int_\Gamma \frac{\partial u(x)}{\partial x_1}U(x-\xi)\cos(\nu,x_1)dx+$$
$$+\int_\Gamma u(x)\frac{\partial U(x-\xi)}{\partial x_2}\cos(\nu,x_2)\,dx-\int_\Gamma \frac{\partial u(x)}{\partial x_2}U(x-\xi)\cos(\nu,x_2)\,dx=$$
$$=\int_D u(x)\Delta_x U(x-\xi)dx=\int_D u(x)\delta(x-\xi)dx=\begin{cases} u(\xi), \ \xi\in D, \\ \frac{1}{2}u(\xi), \ \xi\in\Gamma, \end{cases} \tag{4}$$

so that, ν is the external normal drawn to the boundary Γ of the domain D.

If we multiply the derivative of the fundamental solution with respect to x_1 into the equation and integrate over the domain, we get the following second fundamental relation.

$$\int_\Gamma \frac{\partial u(x)}{\partial x_1}\frac{\partial U(x-\xi)}{\partial x_1}\cos(\nu,x_1)\,dx-\int_\Gamma \frac{\partial u(x)}{\partial x_2}\frac{\partial U(x-\xi)}{\partial x_2}\cos(\nu,x_1)dx+$$
$$+\int_\Gamma \frac{\partial u(x)}{\partial x_1}\frac{\partial U(x-\xi)}{\partial x_2}\cos(\nu,x_2)\,dx+\int_\Gamma \frac{\partial u(x)}{\partial x_2}\frac{\partial U(x-\xi)}{\partial x_1}\cos(\nu,x_2)\,dx=$$
$$=\int_D \frac{\partial u(x)}{\partial x_1}\left[\frac{\partial^2 U(x-\xi)}{\partial x_1^2}+\frac{\partial^2 U(x-\xi)}{\partial x_2^2}\right]dx=$$
$$=\int_D \frac{\partial u(x)}{\partial x_1}\delta(x-\xi)dx=\begin{cases} \frac{\partial u(\xi)}{\partial \xi_1}, \ \xi\in D, \\ \frac{1}{2}\frac{\partial u(\xi)}{\partial \xi_1}, \ \xi\in\Gamma. \end{cases} \tag{5}$$

If we multiply the derivative of the fundamental solution with respect to x_2 into the equation and integrate over the domain, we get the following third fundamental relation.

$$\int_\Gamma \frac{\partial u(x)}{\partial x_1}\frac{\partial U(x-\xi)}{\partial x_2}\cos(\nu,x_1)\,dx+\int_\Gamma \frac{\partial u(x)}{\partial x_2}\frac{\partial U(x-\xi)}{\partial x_1}\cos(\nu,x_1)dx+$$
$$+\int_\Gamma \frac{\partial u(x)}{\partial x_2}\frac{\partial U(x-\xi)}{\partial x_2}\cos(\nu,x_2)\,dx-\int_\Gamma \frac{\partial u(x)}{\partial x_1}\frac{\partial U(x-\xi)}{\partial x_1}\cos(\nu,x_2)\,dx=$$
$$=\int_D \frac{\partial u(x)}{\partial x_2}\left[\frac{\partial^2 U(x-\xi)}{\partial x_1^2}+\frac{\partial^2 U(x-\xi)}{\partial x_2^2}\right]dx=$$
$$=\int_D \frac{\partial u(x)}{\partial x_2}\delta(x-\xi)dx=\begin{cases} \frac{\partial u(\xi)}{\partial \xi_2}, \ \xi\in D, \\ \frac{1}{2}\frac{\partial u(\xi)}{\partial \xi_2}, \ \xi\in\Gamma. \end{cases} \tag{6}$$

Necessary conditions are determined from the received expressions (4),(5),(6).

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FUNCTIONAL ANALYSIS AND THEORY OF OPERATORS

Approximating the common solution of certain nonlinear operator equations using Inertial-Type AA-Viscosity Algorithms

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Many nonlinear problems arising in real-life situations are mathematically modeled as nonlinear operator equations and inclusions. Many iterative algorithms have been proposed in the literature to approximate the solution of certain operator equations. The aim of this talk is to propose an inertial-type AA-viscosity algorithm for approximating the common solutions of certain nonlinear problems such as split variational inclusion problem, generalized equilibrium problem and common fixed point problem involving nonexpansive mappings. Additionally, we demonstrate strong convergence for the proposed method under some mild assumptions, and use our conclusions to approximate the solution of the split feasibility problem, relaxed split feasibility problem, split common null point problem and split minimization problem. Some numerical experiments are also given along with the graphical illustrations and comparison with various currently used approaches from the comparable literature.

Keywords: Viscosity Algorithm, Split Variational Inclusion Problem, Generalized Equilibrium Problem

Application of multistep methods with the third derivative to solve initial-value problem for the ordinary differential equation of the second order

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As was noted above, here investigate the construction of numerical methods with a third derivative and application that solve the following problem:

$$y'' = f(x, y(x), y'(x)\,, \;\; y(x_0) = y_0, \;\; y'(x_0) = y_0' \;\;, x_0 \leq x \leq X. \tag{1}$$

This is an initial-value problem for the ODE's of the second order. As is known the first numerical method for solving the problem (1) was constructed by the Shtörmer. Suppose that problem (1) has a unique continuous solution, which is defined in segment $[x_0, X]$. However, the continuous function $f(x, y, z)$-defined in some closed begin, which has the continuous partial derivatives up to some$P$, inclusively. Let us dived the segment $[x_0, X]$ to N equal part by using the mesh-points $x_{i+1} = x_i + h \quad (i = 0, 1, ...N-1)$, here $0 < h$is the step-size. Moreover, the exact value of the function $y(x)$ at the point $x_i \quad (i = 0, 1, ...N-1)$ let us denote by the $y(x_i)$ and the corresponding approximately value by the $y_i \quad (i = 0, 1, ...N-1)$. By using these denotations one can be constructed some numerical methods for solving the problem (1).

For this aim let's consider the following method:

$$\sum_{i=0}^{k} \alpha_i y_{n+i} = h \sum_{i=0}^{k} \beta_i y'_{n+i} + h^2 \sum_{i=0}^{k} \gamma_i f_{n+i} \tag{2}$$

here $\alpha_i, \beta_i, \gamma_i \ (i = 0, 1, ...k)$ are some real numbers and

$$\alpha_k \neq 0, \;\; f_m = f(x_m, y_m, y'_m) \quad (m = 0, 1, 2, ...).$$

Noted that if known the values$y_0, y_1, ..., y_{k-1}$, then by using method (2) one can find the valuey_k. In this case, the coefficients $\alpha_i, \beta_i, \gamma_i \ (i = 0, 1, ...k)$ considered as known. And now let us consider the application of method (2) to solve the problem (1).

1. Application of Multistep Methods to solve the problem (1)

As was noted above method (2) can be successfully applied to solve the initial-value problem for ODE's of the first order. For this it is enough to put $\gamma_i = 0$ $(i = 0, 1, ..., k)$. It is known that in solving some practical problems, the mathematical model for the named problem independent from the y'. In this case, the method (2) can be presented as:

$$\sum_{i=0}^{k} \alpha_i^{'} y_{n+i} = h^2 \sum_{i=0}^{k} \gamma_i f_{n+i} \tag{3}$$

Which usually Is called as the Shtörmer method. By taking into account that method (3) can be received from method (2) as the particle case, one can say that if method (2) is stable, then method (3) also will be stable. Noted that conception of stability for the method (2) can be given as follows:

Definition 1. Method (2) is called as the stable if the roots of the polynomial $\rho(\lambda) = \alpha_k\lambda^k + \alpha_{k-1}\lambda^{k-1} + ... + \alpha_1\lambda + \lambda_0$ location in the unit circle, on the boundary of which there is not multiply roots.

But the conception of stability for the method of (3) can be presented as:

Definition 2. Method of (3) is called as the stable if the roots of the polynomial $\rho(\lambda) = \alpha_k^{'}\lambda^k + \alpha_{k\cdot 1}^{\backslash}\lambda^{k-1} + ... + \alpha_1\lambda + \lambda_0$ located in the unit Circle on the boundary of which, there is not multiply roots except double root $\lambda = 1$.

It follows from here, that these class methods are separate.

As is known, if the method (3) is stable and has the degree of p, then in the class of methods (3), there are methods with the degree $p \leq 2\,[k/2] + 2$. Here p is the degree for the Multistep Methods of (2) and (3).

By using above describe receive that method (2) is more accurate, than method (3). However, for using method (2), it needs to the definition of values $y_{n+i}^{'}(i = 0, 1, ..., k)$. In other words for using method (2), one must use the method for calculation values $y_{n+i}^{'}(i = 0, 1, ..., k)$. It follows noted, that if the method (2) is stable, then $p \leq 2k + 2$ and if for the calculation of the values of $y_m^{'}$use the following method:

$$\sum_{i=0}^{k} \tilde{\alpha}_i y_{n+i}^{"} = h \sum_{i=0}^{k} \tilde{\beta}_i f_{n+i}, \tag{4}$$

then receive that $p \leq 2\,[k/2] + 2$, for the stable methods.

If take into account the value $y_m^{'}$- calculated by using formula (2) the accuracy of the method (2) will decrease, which is undesirable. Therefore, here suggest using the following method:

$$\sum_{i=0}^{k} \bar{\alpha}_i y_{n+i}^{"} = h \sum_{i=0}^{k} \bar{\beta}_i f_{n+i} + h^2 \sum_{i=0}^{k} \bar{\gamma}_i g_{n+i} \tag{5}$$

here the coefficients $\bar{\alpha}_i, \bar{\beta}_i, \bar{\gamma}_i$ are some real numbers, $\bar{\alpha}_k \neq 0$and the function $g(x,y,y')$ is defined as $g(x,y,y') = f_x^{'}(x,y,y') + f_y^{'}(x,y,y')y'(x) + f_{y'}^{\prime}(x,y,y')f(x,y,y')$.

By using the calculation of the function $g(x,y,y')$one can been constructed the following method:

$$\sum_{i=0}^{k} \alpha_i y_{n+i} = h \sum_{i=0}^{k} \beta_i y_{n+i}^{'} + h^2 \sum_{i=0}^{k} \gamma_i f_{n+i} +$$

$$+ h^3 \sum_{i=0}^{k} l_i g_{n+i} \quad , \; n = 0,1,...,N-k, \quad \alpha_k \neq 0. \tag{6}$$

For the estimation of the order of accuracy of the method (6) let us to consider the following theorem.

Theorem. *If method (6) is stable and has the degree of*p*, then there are methods type (6) with the degree* $p = 3k+4$ *for the*$k = 2m$*. If* $\beta_k = \gamma_k = l_k = 0$ *and method (6) is stable, then there are stable methods with degree* $p = 3k$*.*

For the demonstration the result of this Theorem, let us to consider the following method.

$$y_{n+2} = (y_{n+1} + y_n)/2 + h(31y_{n+1}^{'} - 25y_n^{'})/4 -$$

$$-h^2(63y_{n+1}^{''} + 57y_n^{''})/20 + h^3(233y_{n+1}^{'''} - 97y_n^{'''})/240,$$

Which is stable and has the degree $p = 6$.

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On the convergence properties of positive linear operators including special polynomials given by generating functions method

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Some polynomials -also called special or orthogonal polynomials—have distinctive qualities that make them useful in a variety of mathematical situations, such as operator theory and approximation theory. In approximation theory, where the goal is to identify simpler functions that closely approach complex ones, special polynomials are important. These polynomials are essential building blocks for developing approximation strategies that efficiently approximate datasets or functions.

There are several approaches to figuring out approximation in approximation theory. Korovkin's theorem is the technique utilized for linear positive operators. It is a version of the theorem of Bernstein. It indicates the prerequisites that any function selected from the space of continuous functions must meet in order for it to converge uniformly in a compact region.

Generating functions are an important tool in number theory, combinatorics, and other areas of mathematics. They are just formal power series and encode data about a set of numbers or objects. We can reduce problems involving sequences into problems involving functions by generating functions, which makes it easier to analyze and solve these problems using algebraic and calculus techniques.

In this study, by using the generating function method we introduce a form of Szasz-type positive linear operators involving Euler-type polynomials. We examine the convergence properties of our operators such as the Korovkin-Bohman theorem and evaluate the rate of first-order modulus of continuity, second-order modulus of smoothness, Petree's K-functional, and Lipschitz type space.

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Modeling of multicell colony functioning under programmed cell death (Apoptosis)

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Programmed cell death (PCD) is a genetically regulated process, during which cells destroy themselves via some internal processes [1]. This phenomenon is explainable in multicellular organisms: as damaged cells harm the entire organism, they abolish themselves for the benefit of the organism as a whole. However, this behavior is rather unobvious for unicellular organisms: death of a single cell in that organism means death of the organism itself [2].

The mechanism of kin selection [3] is responsible for PCD in unicellular organisms. The organisms live to benefit both themselves and their kin to the best of their ability. For example, if a colony of yeast encounters lack of necessary nutrition, the PCD is activated. The process continues until a small amount of cells is left that is able to survive hard times without damage and, eventually, restore the colony. Moreover, dead cells can provide additional energetic resources as they can be consumed by healthy cells in the colony [4].

Researching PCD is an actual problem. Insufficient PCD or excessive PCD may lead to various diseases: cancer, AIDS, ischemia, neurodegenerative (Parkinson's, Alzheimer's) ones [5]. Those diseases are important problems of humanity, and there is still no effective cure for them.

For the first time in the present work, we have constructed a model that considers two main paths of apoptosis – external and internal – and takes into account the phenomenon of necrophagocytosis, the ability of dead cells to be consumed by the healthy ones.

The apoptosis, a form of PCD, is an energy-dependent process that includes activating a group of caspases (ferments that destroy protein molecules) that cause cell death. It is modeled with the help of a system of ordinary differential equations under the assumption that a part of deceased cells are consumed by their neighbors.

An assumption is made that there are various cell subpopulations undergoing different stages of the apoptosis. We consider subpopulations of healthy cells Γ_1, apoptotic cells undergoing an external path of the apoptosis Γ_2, apoptic cells undergoing an internal path of the apoptosis Γ_3, apoptotic cells on the final stage of the apoptosis activated by executioner caspase Γ_4 and dead cells Γ_5. The population Γ_i is denoted with a variable x_i, $i = 1, \ldots, 5$.

The system of differential equations that describes the process follows.

$$
\begin{gathered}
\dot{x}_1 = \mu_g f_{gr} x_1 - (\mu_e x_4^\alpha) x_1 - \mu_i \tfrac{1}{1+k_{x_3} S} x_1, \\
\dot{x}_2 = (\mu_e x_4^\alpha) x_1 - k_{x_4} x_2, \ \dot{x}_3 = \mu_i \tfrac{1}{1+k_{x_3} S} x_1 - k_{x_5} x_3, \\
\dot{x}_4 = k_{x_4} x_2 + k_{x_5} x_3 - \mu_d x_4, \ \dot{S} = -k_c(x_1 + x_2 + x_3) + \gamma \mu_d x_4, \\
\mu_g = \mu_L \left(1 - \tfrac{x_1 + x_2 + x_3 + x_4}{X_{\max}}\right), \ f_{gr} = 1 - e^{-\frac{S}{K_S}},
\end{gathered}
$$

where $f_g r$ is a fraction of growing cells, S is available nutrition and K_S is the Tessier constant. μ_g is specific velocity of cell growth, $X_{\max}$ is the maximal amount of cells able to exist in a given volume, $(\mu_e x_4^\alpha)$ is a specific velocity of constructing cells undergoing an external path of the apoptosis. μ_L, μ_e, α, μ_1, k_{x_3}, k_{x_4}, k_{x_5}, μ_d, and k_c are some constant coefficients.

We consider two scenarios. In the first scenario, we study the influence of initial available nutrition on the colony growth. There exist such values of the parameter that first we observe an increase in the amount of healthy cells, but then an intensive growth halts, and the amount of healthy cells tends to a constant amount. Moreover, the amount of cells in subpopulations undergoing different stages of the apoptosis also tends to a constant value. That means that the population of the whole colony tends to a constant, i.e. the biological system tends to an equilibrium state.

In the second scenario, we study the influence of the amount of initially healthy cells on the colony growth. At first, due to the lack of nutrition, the amount of healthy cells starts decreasing. However, due to the mechanisms of apoptosis and necrophagocytosis available nutrition increases and the colony starts growing.

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Self-adjointness of the Schrödinger operator with a spherically symmetric homogeneous potential

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Let us denote by H_a^W the Schrödinger operator acting in $L^2\left(B\left(0,a\right)\right)$ according to the formula

$$H_a^W u = -\triangle u + W\left(r\right) u,$$

with domain

$$D\left(H_a^W\right)=\left\{u\left(\xi\right)\in L^2\left(B\left(0,a\right)\right):\ \left.\frac{\partial u\left(\xi\right)}{\partial r}\right|_{|\xi|=a}=0,\right.$$

$$\left.\forall\varphi\in S^{n-1},\ H_a^W u\in L^2\left(B\left(0,a\right)\right)\right\},$$

where $\xi=(\xi_1,\xi_2,...,\xi_n)$, $B\left(0,a\right)$ is a ball in n-dimensional Euclidean space R^n of radius a with center at the origin, S^{n-1} is $(n-1)$- dimensional unit sphere, i.e.

$$S^{n-1}=\left\{(\eta_1,\eta_2,...,\eta_n)\in R^n:\quad \eta_1^2+\eta_2^2+...+\eta_n^2=1\right\},$$

$r=|\xi|$, $\triangle=\sum\limits_{k=1}^{n}\frac{\partial^2}{\partial\xi_k^2}$ is the classical Laplace operator, $W(r)$ is a sufficiently smooth real homogeneous function, i.e.$W(tr)=t^\gamma W(r)$, $\gamma>-2$.

Our goal in this work is to prove the self-adjointness of the operator H_a^W. Note that this work was motivated by the works of M. Dauge and B. Helffer [1], [2], in which the cases $n=1,2,3$ were studied. Therefore, we will assume that the dimension of the Euclidean space R^n is greater than three.

Theorem. *Let the dimension of the Euclidean space R^n be greater than three. Then the operator H_a^W is self-adjoint in $L^2\left(B\left(0,a\right)\right)$.*

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Solvability of a boundary value problem for a second order elliptic differential-operator equation with a quadratic complex parameter

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In the paper in a separable Hilbert space H, we study the solvability of the following boundary value problem :

$$L(\lambda)u := \lambda^2 u(x) - u''(x) + Au(x) = f(x),\ x \in (0,1), \tag{1}$$

$$\begin{aligned} L_1(\lambda)u &:= \lambda u'(0) + u(1) = f_1, \\ L_2 u &:= u(0) = f_2, \end{aligned} \tag{2}$$

where λ is a complex parameter ; A is f positive operator in H.

Definition 1. A linear closed operator A in Hilbert space H is said to be f positive, if its domain of defitition D(A) is dense in H and for some $f \in [0,p)$, for all the points $\mu \in \not{C}$, from the angle $|\arg\mu| \leq \varphi$ (including $\mu = 0$) there exist the operators $(A+\mu I)^{-1}$, for which with these , we have the estimation

$$\left\|(A+\mu I)^{-1}\right\|_{B(H)} \leq C(1+|\mu|)^{-1},$$

where I is a unit operator in H, $C = const > 0$.

The simplest example of φ -positive operators is a self-adjoint, positive –definite operator acting in Hilbert space.

Definition 2. (see.[1], Theorem 1.14.5). Let A be a f is positive operator in Hilbert space H. Then for $\theta \in (0,1)$, $p > 1$, $n \in N$ the interpolation space $(H(A^n), H)_{\theta,p}$ of Hilbert spaces $H(A^n)$ and H is determined by the equality

$$(H(A^n),\, H)_{\theta,p} := \left\{ u : u \in H,\ \|u\|_{(H(A^n),H)_{\theta,p}} := \right.$$

$$\left. := \left(\int_0^{+\infty} t^{-1+n\theta p} \left\|A^n e^{-tA} u\right\|_H^P dt \right)^{\frac{1}{p}} < \infty \right\}.$$

Whererein, by the definition $(H(A^n),\, H)_{0,p} := H(A^n),\ \ (H(A^n),\, H)_{1,p} := H.$

At first in the space Hwe consider the following boundary –value problem:

$$L\left(\lambda\right)u := \lambda^2 u\left(x\right) - u''\left(x\right) + Au\left(x\right) = 0, x \in \left(0,\, 1\right), \tag{3}$$

$$\begin{aligned} &L_1\left(\lambda\right)u := \lambda u'\left(0\right) + u\left(1\right) = f_1\,, \\ &L_2 u := u\left(0\right) = f_2. \end{aligned} \tag{4}$$

Theorem 1. Let A be φ positive operator in H. Then, for $f_k \in (H(A^2), H)_{1-\frac{k}{2}+\frac{1}{4p},p}$, $k = 1, 2,,$ $p \in (1, +\infty)$ and for rather large $|\lambda|$ from the angle $|\arg\lambda| \leq \varphi < \frac{\pi}{2}$ the problem (3),(4) has a unique solution $u \in W_p^2\left((0,1)\ ;\ H\left(A\right),\, H\right)$ and for the solution we have the following non coercive estimation

$$|\lambda|^{\,2} \left\|u\right\|_{L_p((0,1);H)} + \left\|u''\right\|_{L_p((0,1);H)} + \left\|Au\right\|_{L_p((0,1);H)} \leq$$

$$\leq C \sum_{k=1}^{2} \left(\left\|f_k\right\|_{(\,H(A^2),\, H)_{1-\,\frac{k}{2}+\frac{1}{4p},p}} + |\lambda|^{k-\frac{1}{2p}} \left\|f_k\right\|_H \right).$$

Theorem 2. Let A be φ is positive operator in H.

Then for $f \in L_p\left((0,1);\, H\left(A\right)\right),\ \ f_k \in \left(H\left(A^2\right), H\right)_{1-\frac{k}{2}+\frac{1}{4p},p},\ \ \ k = 1, 2,\, p \in (1, +\infty)$ and for rather large $|\lambda|$ from the angle $|\arg\lambda| \leq \varphi < \frac{\pi}{2}$, the problem (1),(2) has a unique solution $u \in W_p^2\left((0,1)\, ; H\left(A\right), H\right)$ and for the solution we have the following noncoercive estimation

$$|\lambda|^2 \left\|u\right\|_{L_p((0,1);H)} + \left\|u''\right\|_{L_p((0,1);H)} + \left\|Au\right\|_{L_p((0,1);H)} \leq$$

$$\leq C \left[|\lambda|^{3/2} \left\|f\right\|_{L_p((0,1);H(A))} + \sum_{k=1}^{2} \left(\left\|f_k\right\|_{(H(A^2),H)_{1-\frac{k}{2}+\frac{1}{4p},p}} + |\lambda|^{k-\frac{1}{2p}} \left\|f_k\right\|_H \right) \right].$$

Note that a boundary value problem for the equation (1) in the case when the coefficients in boundary conditions are complex numbers, and the boundary conditions are regular in the sence of Birkhoff-Tamarkin, was first studied in the monograph [2, ch.5, §5.4], where coercive solvability of the considered boundary value problems in the space $L_p\left((0,1);\, H\right),\ \ p \in (1, +\infty)$, is proved with respect to u.

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A new method of construction of a system of Chebyshev-Hermite polynomials based on the weighted function

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In this thesis considers investigating Chebyshev-Hermite polynomials and the aim of this thesis is to show a new method for constructing the Chebyshev-Hermite polynomials system. For this purpose, various coefficients of orthogonal polynomials were determined associated with the weight function and as a result, the norm was found. Let's determine several coefficients of orthogonal and orthonormal polynomials and consider the appropriate weight function of these polynomials. For this, we will use the existence condition.

Definition 1. Let $\{P_n(x)\}_{n=0}^{\infty}$ be a system of polynomials, where every polynomial $P_n(x)$ has the degree n. If for all polynomials of this system $\int_a^b h\,(x)\,P_n(x)P_m(x)\ dx = 0, n \neq m$ then the polynomials $\{P(x)\}_{n=1}^{\infty}$ called orthogonal in $(a,\ b)$ with respect to the weight function $h(x)$.

If moreover $\|P_n(x)\|_{h(x)} = \left(\int_a^b h(x)P_n^2(x)dx\right)^{\frac{1}{2}} = 1$ for every $n\ =\ 0,\ 1,\ 2,\ \ldots,$ then the polynomials are called orthonormal in (a, b). So the condition of the orthonormality of the system $\{P_n\,(x)\}_{n=0}^{\infty}$ has the form
$\int_a^b h\,(x)\,P_n\,(x)\,P_m\,(x)\,dx = \delta_{mn}$, where δ_{mn} is Kronecker delta which is defined by $\delta_{mn} = \begin{Bmatrix} 0, m \neq n \\ 1, m = n \end{Bmatrix}$for $m, n = \{0, 1, 2\ldots\}$.The Chebyshev-Hermite polynomials $H_n\,(x) = (-1)^n\,e^{x^2}(e^{-x^2})^{(n)}$ are orthogonal on the interval $(-\infty, \infty)$ with

respect to weighted function $h(x) = e^{-x^2}$the normal distribution and let us clearly define several coefficient of these polynomials:

$H_0(x) = 1, H_1(x) = \ 2x, H_2(x) = 4x^2 - \ 2, H_3(x) = 8x^3 - \ 12x, \ldots$. Lets show that these $H_0(x)\,,\ H_1(x)\,,\ H_2(x)\,,\ H_3(x)$ polynomials are orthogonal polynomials with respect to weighted function

$\int_{-\infty}^{+\infty} h(x) H_0(x) H_1(x)\, dx = \int_{-\infty}^{+\infty} e^{-x^2} 2x dx \ = - \ e^{-x^2}|_{-\infty}^{+\infty} = 0$

$\int_{-\infty}^{+\infty} h(x) H_0(x) H_2(x)\, dx = \int_{-\infty}^{+\infty} e^{-x^2} (4x^2 - 2)\, dx = 0$

$\int_{-\infty}^{+\infty} h(x) H_0(x) H_3(x)\, dx = \int_{-\infty}^{+\infty} e^{-x^2} (8x^3 - 12x)\, dx = 0$

……………..

Using the above orthogonality condition, let's define an orthonormal polynomial with respect to the weight function.

$\int_{-\infty}^{+\infty} h(x) H_0^2(x)\, dx = \int_{-\infty}^{+\infty} e^{-x^2} dx = \sqrt{\pi}$,norm is $\mu_0 = \sqrt{\frac{1}{\sqrt{\pi}}}$

$\int_{-\infty}^{+\infty} h(x) H_1^2(x)\, dx = 2\sqrt{\pi}$,norm is $\mu_1 = \sqrt{\frac{2}{\sqrt{\pi}}}$

$\int_{-\infty}^{+\infty} h(x) H_2^2(x)\, dx = 2!2^2\sqrt{\pi}$ norm is $\mu_2 = \sqrt{\frac{2^2}{2\sqrt{\pi}}}$

$\int_{-\infty}^{+\infty} h(x) H_3^2(x)\, dx = 3!2^3\sqrt{\pi}$ norm is $\mu_3 = \sqrt{\frac{2^3}{3!\sqrt{\pi}}}$

……………..

Now let us define some coefficients of the system of orthonormal polynomials using a new method with the help of the method of mathematical induction as follows $\mu_n = \sqrt{\frac{2^n}{n!\sqrt{\pi}}}$. So, the orthonormal Chebyshev-Hermite polynomial is generally as follows. Consequently we get

$H_0(x) = \ \frac{H_n(x)}{\sqrt{n!2^n\sqrt{\pi}}} = \ \frac{(-1)^n}{\sqrt{n!2^n\sqrt{\pi}}} e^{x^2} \left(e^{-x^2}\right)^n$

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Nodal solutions of some nonlinear boundary value problems for fourth-order ordinary differential equations

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We consider the following nonlinear boundary value problem

$$(p(x)y''(x))'' - (q(x)y'(x))' = \tau r(x)f(y(x)),\ x \in (0,l), \tag{1}$$

$$y'(0)\cos\alpha - (py'')(0)\sin\alpha = 0, \tag{2}$$

$$y(0)\cos\beta + Ty(0)\sin\beta = 0, \tag{3}$$

$$y'(l)\cos\gamma + (py'')(l)\sin\gamma = 0, \tag{4}$$

$$(a\lambda + b)y(l) - (c\lambda + d)Ty(l) = 0, \tag{5}$$

where $Ty \equiv (py'')' - qy'$, $p \in C^2([0,l];(0,+\infty))$, $q \in C^1([0,l];[0,+\infty))$, τ is a positive number, $r(x) \in C([0,l];(0,+\infty))$, α, β, γ, a, b, c, d are real constants such that

$$\alpha, \beta, \gamma \in [0, \pi/2] \text{ and } \sigma = bc - ad > 0.$$

The nonlinear term $f \in C(\mathbb{R};\mathbb{R})$ satisfy the following conditions:

$$sf(s) > 0 \text{ for } s \in \mathbb{R},\ s \neq 0; \tag{6}$$

$$\underline{f}_0,\ \overline{f_0},\ \underline{f}_\infty,\ \overline{f}_\infty \in (0,+\infty) \text{ with } \underline{f}_0 \neq \overline{f}_0,\ \underline{f}_\infty \neq \overline{f}_\infty, \tag{7}$$

where

$$\underline{f}_0 = \liminf_{|s|\to 0} \frac{f(s)}{s}\,,\ \overline{f}_0 = \limsup_{|s|\to 0} \frac{f(s)}{s}\,, \tag{8}$$

$$\underline{f}_\infty = \liminf_{|s|\to +\infty} \frac{f(s)}{s}\,,\ \overline{f}_\infty = \limsup_{|s|\to +\infty} \frac{f(s)}{s}\,; \tag{9}$$

In this note we determine the interval for r in which there exist solutions of problem (1)-(5) with a fixed number of simple nodal zeros.

We denote by BC_0 and BC_λ the sets of functions that satisfy the boundary conditions (2)-(4) and (2)-(5), respectively.

Let $E = C^3[0,l] \cap BC_0$ be a Banach space which is equipped with the usual norm

$$||y||_3 = \sum_{s=0}^{3} ||\, y^{(s)}||_\infty,$$

where

$$||y||_\infty = \max_{x\in[\,0,\,l]} |y\,(x)|.$$

In [1] using the Prüfer type transformation, the author constructed classes $\mathcal{S}_k^\nu$, $k \in \mathbb{N}$, $\nu \in \{+\,,\,-\}$, of functions $y \in E$, which a fixed number of simple nodal zeros. The sets $\mathcal{S}_k^+$ and $\mathcal{S}_k^-$ are pairwise disjoint open subsets of E.

We consider the following linear eigenvalue problem

$$\begin{cases} \ell(y)(x) = \lambda r(x) y(x),\ x \in (0,l), \\ y \in BC_\lambda. \end{cases} \tag{10}$$

Problem (10) was studied in [2], where it was established that the eigenvalues of this problem are real and simple, and form an infinitely increasing sequence $\{\lambda_k\}_{k=1}^\infty$ such that $\lambda_k > 0$ for $k \geq 3$; in this case, each of the first two eigenvalues can be either positive or non-positive. Moreover, by [2, Theorem 2.2] it follows from [1, Remark 2.1] and [3, § 3.1] that for each $k \in \mathbb{N}$ the eigenfunction $y_k(x)$ corresponding to the eigenvalue $\lambda_k > 0$ lies in S_k (in this case the function $y_k(x)$ has either $k-1$ or $k-2$ simple nodal zeros in $(0,l)$).

Throughout what follows we will assume that the first eigenvalue of problem (10) is positive.

Theorem 1. *Let conditions (6)-(9) be satisfied and for some $k \in \mathbb{N}$ one of the following conditions holds:*

$$\frac{\lambda_k}{\underline{f}_0} < \tau < \frac{\lambda_k}{\overline{f}_\infty};\ \ \frac{\lambda_k}{\underline{f}_\infty} < \tau < \frac{\lambda_k}{\overline{f}_0}\,.$$

Then problem (1)-(5) has two solutions y_k^+ and y_k^- such that $y_k^+ \in \mathcal{S}_k^+$ and $y_k^- \in \mathcal{S}_k^-$, and consequently, y_k^+ has $k-1$ or $k-2$ simple zeros in $(0,l)$ and is positive near $x = 0$, and y_k^- has either $k-1$ or $k-2$ simple zeros in $(0,l)$ and is negative near $x = 0$.

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On convergence of spectral expansion in the eigenfunctions of a third-order differential operator

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On the interval $G = (0,1)$, consider the differential operator

$$Lu = u^{(3)} + P_2(x)u^{(1)} + P_3(x)u$$

with coefficients $P_l(x) \in L_1(G), \ \ l = \overline{2,3}$.

By $D\,(G)$ we denote the class of functions absolutely continuous together with their derivatives of order less than or equal to two on the interval $\overline{G} = [0,1]$.

An eigenfunction of the operator L corresponding to the eigenvalue λ understood a function $u(x) \in D(G)$ that is not identically zero and satisfies the equation $Lu + \lambda u = 0$ almost everywhere in G the equation (see [1]).

Let $\{u_k(x)\}_{k=1}^{\infty}$ be a complete orthonormed in $L_2\,(G)$ system consisting of eigenfunctions of the operator L, and let while $\{\lambda_k\}_{k=1}^{\infty}$ be the corresponding system of eigenvalues, $Re\lambda_k = 0$.

By $W_p^1(G)$, $p \geq 1$, we denote the class of functions $f(x)$ absolutely continuous on the interval $\overline{G}$ for which $f'\,(x) \in L\,_p(G)$.

We write $\mu_k = (-i\lambda_k)^{1/3}$, $Im\lambda_k \geq 0$; $\mu_k = (i\lambda_k)^{1/3}$, $Im\lambda_k < 0$, and introduce a partial sum of the orthogonal expansion of the function $f(x) \in W_1^1(G)$ with respect to the system $\{u_k(x)\}_{k=1}^{\infty}$:

$$\sigma_\nu(x, f) = \sum_{\mu_k \leq \nu} f_k u_k(x)\,,\ \ \nu > 0,$$

where $f_k = (f, u_k) = \int_0^1 f(x)\overline{u_k(x)}dx$.

Study the behavior of the difference $R_\nu(x, f) = f(x) - \sigma_\nu(x, f)$. In this paper we prove the following results.

Theorem. *Assume that a function* $f(x) \in W_1^1(G)$ and system $\{u_k(x)\}_{k=1}^{\infty}$ satisfy the conditions

$$\left| f(x)\,\overline{u_k^{(2)}}(x)\,\Big|_0^1 \right| \leq C(f)\,\mu_k^{\alpha}\,\|u_k\|_{\infty}\,,\quad 0 \leq \alpha < 2,\ \ \mu_k \geq 1 \tag{1}$$

$$\sum_{k=2}^{\infty} k^{-1}\,\omega_1\left(f', k^{-1}\right) < \infty. \tag{2}$$

Then the spectral expansion of the $f(x)$ with respect to the system $\{u_k(x)\}_{k=1}^{\infty}$ converges absolutely and uniformly on the interval $\overline{G}$, and the estimate

$$\|\,R_\nu\,(\,\cdot\,,\,f)\,\|_{\,C\,[0,1\,]} \leq$$

$$\leq const\left\{C\,(\,f\,)\;\nu^{\,\alpha-2} + \omega_1\,(f',\,\nu^{-1}) + \sum_{k=[\nu]}^{\infty} k^{-1}\omega_1\,(f',\,k^{-1}) + \right.$$
$$\left. +\;\nu^{\,-1}\left(\|f'\|_1 + (\|\,f\|_{\,\infty} + \|\,f'\,\|_{\,p})\sum_{l\,=\,2}^{3}\nu^{\,2-l}\,\|\,P_{\,l}\,\|_{\,1}\right)\right\}$$

holds where $\nu \geq \nu_0 = 8\pi$, $\omega\,(\cdot,\delta)$is the modulus of continuity on the space $L_1\,(G)\,,\ \|\cdot\|_{\,p} = \|\cdot\|_{L_p(G),}$ and const is independent of the function $f(x)$.

Corollary 1. If the function $f(x) \in W_1^1(G)$ in Theorem satisfies the conditions $f(0) = f\,(1) = 0$, then condition (1) is necessarily satisfied (with the $C\,(f) = 0$), then the spectral expansion of function $f(x)$ with respect to the system $\{u_k(x)\}_{k=1}^{\infty}$ converges absolutely and uniformly on the interval $\overline{G}$, and the following estimate holds:

$$\|\,R_\nu\,(\,\cdot\,,\,f)\,\|_{\,C\,[0,1\,]} \leq$$

$$\leq const\left\{\omega_1\left(f,\nu^{-1}\right)+\sum_{k=[\nu]}^{\infty}k^{-1}\omega_1\left(f',k^{-1}\right)+\nu^{-1}\left\|f'\right\|_1\left(1+\sum_{l=2}^{3}\nu^{2-l}\left\|P_l\right\|_1\right)\right\},$$

Corollary 2. If the function $f(x) \in W_1^1(G)$ in Theorem satisfies the relations $f(0) = f(1) = 0$ and $f'(x) \in H_1^{\beta}(G)$, $0 < \beta \leq 1$ ($H_1^{\beta}(G)$ is the Nikolski), then conditions (1) and (2) are necessarily satisfied, then the spectral expansion of function $f(x)$ converges absolutely and uniformly on the interval $\overline{G} = [0,1]$, and the following estimate holds:

$$\| R_\nu(\cdot, f) \|_{C[0,1]} \leq const\ \nu^{-\beta} \|f'\|_1^{\beta}, \qquad \nu \geq \nu_0$$

where $\|f'\|_1^{\beta} = \|f'\|_1 + \sup_{\delta>0}\delta^{-\beta}\omega_1(f',\delta)$ and the const independent of the function $f(x)$.

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The absolute deferred Riesz summability of factored infinite series

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Let (p_n) be a sequence of positive real numbers and let $a = (a_n)$ and $b = (b_n)$ be sequences of nonnegative integers with the conditions $a_n < b_n$ for $n = 1, 2, 3, ...$ and $\lim_{n\to\infty} b_n = \infty$, such that

$$P_{a_n+1}^{b_n} = \sum_{k=a_n+1}^{b_n} p_k \neq 0.$$

Assume that $\sum d_n$ is a given infinite series with partial sums (s_n). Then the sequence-to-sequence transformation

$$D_a^b R_n = \frac{1}{P_{a_n+1}^{b_n}} \sum_{m=a_n+1}^{b_n} p_m s_m$$

defines the sequence $D_a^b R_n$ of the deferred Riesz mean of the sequence (s_n), generated by the sequence of coefficients $(p_n)[1], [2]$.

The series $\sum d_n$ is said to be absolute deferred Riesz summable if

$$\sum_{n=1}^{\infty} \theta_n^{k-1} |D_a^b R_n - D_a^b R_{n-1}| < \infty,$$

where (θ_n) is any sequence of positive numbers.

In this study, taking into account of this method, the result given in [1] on the absolute Riesz summability factors of infinite series has been generalized for the deferred Riesz summability method.

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On the Riesz submethod summability of factored infinite series

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Let F be an infinite subset of $\mathbf{N}$ as the range of a strictly increasing sequence of positive integers, with $F = (\lambda(n))_{n=1}^{\infty}$. Let (p_n) be a sequence of positive real numbers such that

$$P_{\lambda(n)} = p_0 + p_1 + p_2 + ... + p_{\lambda(n)} \neq 0 \ \ (n \geq 0),$$

and by convention, $p_{-1} = P_{-1} = 0$.

Assume that $\sum a_n$ is a given infinite series with partial sums (s_n). Then the transformation R_n^{λ} is defined as

$$R_n^{\lambda} = \frac{1}{P_{\lambda(n)}} \sum_{m=0}^{\lambda(n)} p_m s_m \ \ [2].$$

The series $\sum a_n$ is said to be absolute $|R_n^{\lambda}, p_n, \theta_n|_k, k \geq 1$, summable if

$$\sum_{n=1}^{\infty} \theta_n^{k-1} |R_n^{\lambda} - R_{n-1}^{\lambda}| < \infty,$$

where (θ_n) is any sequence of positive numbers.

According to this method, we obtain the generalization of the result given in [1] on the absolute Riesz summability factors of infinite series.

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The regularized trace formula for a discontinuous Sturm-Liouville problem with a spectral parameter dependent jump condition

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We consider the boundary value problem

$$-y'' + q(x)y = \lambda y, \quad x\epsilon(-1,0) \cup (0,1), \tag{1}$$

$$\left.\begin{array}{r} y(-1) = y(1) = 0, \\ y(-0) = y(+0), \\ y'(-0) - y'(+0) = \lambda m y(0), \end{array}\right\} \tag{2}$$

where $q(x) \in W_2^1(-1,1)$ is a complex-valued function, $m \neq 0$ is a complex number. The aim of the paper is to calculate the first regularized trace for the problem (1), (2).

One can prove the following asymptotic expansion of the characteristic function $\Delta(\lambda)$ of problem (1), (2), which is sharper than the expansion of the same $\Delta(\lambda)$ obtained in [1], where $q(x)$ was assume to be a summable function on $(-1,1)$:

$$\Delta\left(\rho^2\right)=[-m\rho^2+2q_2(0)-mq_{30}(0)+2q_1(0)+\frac{1}{\rho^2}q_1(0)q_{30}(0)-$$
$$-mq_{20}(0)+\frac{1}{\rho^2}q_2(0)q_{20}(0)-\frac{m}{\rho^2}q_{20}(0)q_{30}(0)+mq_1(0)q_2(0)-\frac{1}{\rho^2}q_{31}(0)q_1(0)-$$
$$-\frac{1}{\rho^2}q_2(0)q_{21}(0)]\sin^2\rho+[2\rho+\rho mq_2(0)-\frac{1}{\rho}q_{31}(0)+\frac{1}{\rho}q_{30}(0)+\rho mq_1(0)-$$
$$-\frac{2}{\rho}q_1(0)q_2(0)+\frac{m}{\rho}q_{30}(0)q_1(0)+\frac{1}{\rho}q_{20}(0)-\frac{1}{\rho}q_{21}(0)+\frac{m}{\rho}q_{20}(0)q_2(0)]sin\rho\cos\rho+$$
$$+[-\frac{1}{4\rho}\int_0^1 q'(t)\cos\rho(2t-1)dt+\frac{m}{4}\int_0^1 q'(t)\sin\rho(2t-1)dt-$$
$$-\frac{1}{2\rho}\int_0^1 q(t)q_2(t)\cos\rho(2t-1)dt-\frac{1}{2\rho}\int_{-1}^0 q(t)q_1(t)\cos\rho(2t+1)dt-$$
$$-\frac{m}{2}\int_{-1}^0 q(t)q_1(t)\sin\rho(2t+1)dt+\frac{1}{4\rho}\int_{-1}^0 q'(t)\cos\rho(2t+1)dt]\sin\rho+$$
$$+[-\frac{1}{4\rho}\int_0^1 q'(t)\sin\rho(2t-1)dt+q_1(0)\frac{1}{4\rho}\int_0^1 q'(t)\cos\rho(2t-1)dt-$$
$$-\frac{m}{4\rho}q_1(0)\int_0^1 q'(t)\sin\rho(2t-1)dt+\frac{1}{4\rho}\int_{-1}^0 q'(t)\sin\rho(2t+1)dt-$$
$$-\frac{1}{2\rho}\int_0^1 q_2(t)q(t)\sin\rho(2t-1)dt+\frac{1}{2\rho}\int_{-1}^0 q_1(t)q(t)\sin\rho(2t+1)dt+$$
$$+\frac{m}{2\rho}q_2(0)\int_{-1}^0 q_1(t)q(t)\sin\rho(2t+1)dt]cos\rho+$$
$$+(-q_2(0)-q_1(0)-mq_1(0)q_2(0))+O\left(\frac{e^{|\tau|}}{\rho^2}\right). \qquad (3)$$

The following theorem was proved in [1].

Theorem. *Let $q(x)$ be a complex-valued function, summable on $[-1,1]$ and $d := 4 + (mq_2(0))^2 + (mq_1(0))^2 - 2m^2 q_2(0)q_1(0)$. Then, the spectrum of problem (1), (2) consists of two sequences of eigenvalues $\lambda_{1,n} = \rho_{1,n}^2$, $n = 0, 1, \dots,$ and $\lambda_{2,n} = \rho_{2,n}^2$, $n = 1, 2, \dots,$ counted with their algebraic multiplicities, for which the following asymptotic equalities hold:*

$$\rho_{1,n} = \pi n + \frac{\alpha_1}{n} + o\left(\frac{1}{n}\right) \quad \textit{and} \quad \rho_{2,n} = \pi n + \frac{\alpha_2}{n} + o\left(\frac{1}{n}\right), \tag{4}$$

where α_1 and α_2 are defined by the following formulas:

$$\alpha_1 = \frac{-\left(2 + mq_2(0) + mq_1(0)\right) + \sqrt{d}}{-2m\pi},$$

$$\alpha_2 = \frac{-\left(2 + mq_2(0) + mq_1(0)\right) - \sqrt{d}}{-2m\pi},$$

$$0 \le arg\sqrt{d} < \pi.$$

If $d \neq 0$, then the eigenvalues of the problem are asymptotically simple.

From (3) it follows that if $q(x) \in W_2^1(-1,1)$, then the following asymptotics, that are more precise than the asymptotics (4), are true:

$$\rho_{1,n} = \pi n + \frac{\alpha_1}{n} + \frac{\xi_n^{(1)}}{n^2}, \quad \rho_{2,n} = \pi n + \frac{\alpha_2}{n} + \frac{\xi_n^{(2)}}{n^2}, \quad \left\{\xi_n^{(1)}\right\}, \left\{\xi_n^{(2)}\right\} \in l_2.$$

Then

$$\lambda_{1,n} = (\rho_{1,n})^2 = \pi^2 n^2 + 2\pi\alpha_1 + \frac{\eta_n^{(1)}}{n}, \quad \left\{\eta_n^{(1)}\right\} \in l_2, \tag{5}$$

$$\lambda_{2,n} = (\rho_{2,n})^2 = \pi^2 n^2 + 2\pi\alpha_2 + \frac{\eta_n^{(2)}}{n}, \quad \left\{\eta_n^{(2)}\right\} \in l_2.$$

Therefore, the sum

$$S_\lambda := \sum_{n=0}^{\infty} \left(\lambda_{1,n} - \pi^2 n^2 - 2\pi\alpha_1\right) + \sum_{n=1}^{\infty} \left(\lambda_{2,n} - \pi^2 n^2 - 2\pi\alpha_2\right) \tag{6}$$

is convergent (more precisely, both of the above series are convergent). The sum (6) is called *the first regularized trace* for the problem (1), (2).

Theorem. *Let $q(x)$ be a complex-valued function in $W_2^1(-1,1)$ and $\{\lambda_{1,n}\}_{n=0}^{\infty} \cup \{\lambda_{2,n}\}_{n=1}^{\infty}$ be its eigenvalues, counted with their algebraic multiplicities. The following first regularized trace formula holds:*

$$S_\lambda = \sum_{n=0}^{\infty}\left(\lambda_{1,n} - \pi^2 n^2 - 2\pi\alpha_1\right) + \sum_{n=1}^{\infty}\left(\lambda_{2,n} - \pi^2 n^2 - 2\pi\alpha_2\right) =$$

$$= -\pi\left(\alpha_1 + \alpha_2\right) - \pi^2\left(\alpha_1^2 + \alpha_2^2\right) - \pi^2\alpha_1\alpha_2 -$$

$$-\frac{1}{m}\int_{-1}^{1} q(t)dt - \frac{1}{4}\int_{0}^{1} q(t)dt \cdot \int_{-1}^{0} q(t)dt + \frac{1}{4}\left(2q(0) + q(-1) + q(1)\right) -$$

$$-\frac{1}{8}\left(\left(\int_0^1 q(t)dt\right)^2 + \left(\int_{-1}^0 q(t)dt\right)^2\right).$$

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Commutators of two-weighted Φ-admissible singular operators on generalized weighted Orlicz-Morrey spaces

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In this presentation we find sufficient conditions on functions φ_1, φ_2 and weights ω_1, ω_2, $(\omega_1, \omega_2) \in A_{\Phi\Phi}(\mathbb{R}^n)$, with Young function Φ which provide the boundedness of two-weighted Φ-admissible singular operators from one generalized weighted Orlicz-Morrey spaces $M_{\Phi,\varphi_1,\omega_1}(\mathbb{R}^n)$ to $M_{\Phi,\varphi_2,\omega_2}(\mathbb{R}^n)$, and the boundedness of the two-weighted admissible commutators from $M_{\Phi,\varphi_1,\omega_1}(\mathbb{R}^n)$

to $M_{\Phi,\varphi_2,\omega_2}(\mathbb{R}^n)$, where $\omega_1, \omega_2 \in A_\Phi(\mathbb{R}^n)$. Also we find sufficient conditions on functions φ_1, φ_2 and weights ω_1, ω_2, $(\omega_1, \omega_2) \in A_{\Phi\Phi}(\mathbb{R}^n)$, for the boundedness of aforementioned operators in vanishing generalized weighted Orlicz-Morrey spaces $VM_{\Phi,\varphi_1,\omega}(\mathbb{R}^n)$. As applications, we give apriori estimates in generalized weighted Orlicz-Morrey spaces, by considering a special type of Dirichlet problem.

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Weighted adjoint fractional Hardy operators in local generalized Orlicz-Morrey spaces

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We consider the following weighted adjoint fractional Hardy operators

$$\mathcal{A}_\omega^\alpha f(x) = |x|^\alpha \omega(|x|) \int_{|y|\le|x|} \frac{f(y)}{|y|^n \omega(|y|)} dy$$

where $\alpha \ge 0$ and ω is a weight.

A function $\Phi : [0, +\infty] \to [0, +\infty]$ is called a Young function if Φ is convex, left-continuous, $\lim\limits_{r\to+0} \Phi(r) = \Phi(0) = 0$ and $\lim\limits_{r\to+\infty} \Phi(r) = \Phi(+\infty) = +\infty$. From the convexity and $\Phi(0) = 0$ it follows that any Young function is increasing. If there exists $s \in (0, +\infty)$ such that $\Phi(s) = +\infty$, then $\Phi(r) = +\infty$ for $r \geq s$.

For a Young function Φ, the set

$$L_\Phi(\mathbb{R}^n) = \left\{ f \in L_1^{loc}(\mathbb{R}^n) : \int_{\mathbb{R}^n} \Phi(k|f(x)|)dx < +\infty \text{ for some } k > 0 \right\}$$

is called Orlicz space. The space $L_\Phi^{\rm loc}(\mathbb{R}^n)$ endowed with the natural topology is defined as the set of all functions f such that $f\chi_{_B} \in L_\Phi(\mathbb{R}^n))$ for all balls $B \subset \mathbb{R}^n$.

For a Young function Φ and $0 \leq \lambda \leq n$, we denote by $M_{\Phi,\varphi}^{0,loc}(\mathbb{R}^n)$ the local generalized Orlicz-Morrey space, defined as the space of all functions $L_\Phi^{loc}(\mathbb{R}^n)$ with finite quasinorm

$$\|f\|_{M_{\Phi,\varphi}^{0,loc}} = \sup_{r>0} \varphi(r)^{-1}\Phi^{-1}(r^{-n})\|f\|_{L_\Phi(B(0,r))}.$$

In the following two theorems we prove the boundedness of weighted adjoint fractional Hardy operators from one local generalized Orlicz-Morrey space $M_{\Phi,\varphi_1}^{0,loc}(\mathbb{R}^n)$ to another local generalized Orlicz-Morrey space $M_{\Psi,\varphi_2}^{0,loc}(\mathbb{R}^n)$.

Theorem 1. *Let Φ, Ψ be Young functions, $0 < \alpha < n$, $\frac{r^\beta}{\omega(r)} \leq C\frac{t^\beta}{\omega(t)}$, $r^\alpha\varphi_1(r)\Phi^{-1}\left(r^{-n}\right) \leq Ct^\alpha\varphi_1(t)\Phi^{-1}\left(t^{-n}\right)$, $0 < r < t$, $\beta \in \mathbb{R}$ and the functions (φ_1, φ_2) and (Φ, Ψ) satisfy the conditions*

$$\int_r^\infty \frac{\varphi_1(t)\Phi^{-1}\big(t^{-n}\big)}{\omega(t)}\frac{dt}{t} \leq C\frac{\varphi_1(r)\Phi^{-1}\big(r^{-n}\big)}{\omega(r)}$$

and

$$\int\limits_0^r \frac{t^\alpha\varphi_1(t)\Phi^{-1}\left(t^{-n}\right)}{\Psi^{-1}\left(t^{-n}\right)}\frac{dt}{t} \leq C\frac{\varphi_2(r)}{\Psi^{-1}\left(r^{-n}\right)},$$

where C does not depend on r. Then the weighted adjoint fractional Hardy operator $\mathcal{A}_\omega^\alpha$ is bounded from $M_{\Phi,\varphi_1}^{0,loc}(\mathbb{R}^n)$ to $M_{\Psi,\varphi_2}^{0,loc}(\mathbb{R}^n)$.

On the compactness of commutators for the Riesz potential in local Morrey-type spaces

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The main purpose of this paper is to find sufficient conditions for the compactness of commutators operators $[b, I_\alpha]$ on the Local Morrey-type space $LM_{p\theta}^{w(\cdot)}(\mathbb{R}^n)$.

We consider the Riesz potential. Let $f \in L^1_{loc}$. The Riesz potential $I_\alpha(f)$ is defined of

$$(I_\alpha f)(x) = \int\limits_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy, \quad 0 < \alpha < n.$$

The operator I_α plays an important role in the harmonic analysis and in the theory of operators.

The commutators generated by $b \in L^1_{loc}(\mathbb{R}^n)$ and the operator I_α are defined by

$$[b, I_\alpha]\, f(x) = M_b I_\alpha - I_\alpha M_b = \int\limits_{\mathbb{R}^n} \frac{[b(x) - b(y)]\, f(y)}{|x-y|^{n-\alpha}} dy.$$

It is said that the function $b(x) \in L_1^{loc}(\mathbb{R}^n)$ belongs to the space $BMO(\mathbb{R}^n)$, if

$$\|b\|_* = \sup_{x \in \mathbb{R}^n, \quad r>0} \frac{1}{B(x,r)} \int\limits_{B(x,r)} \left| b(y) - b_{B(x,r)} \right| dy < \infty,$$

where

$$b_{B(x,r)} = \frac{1}{|B(x,r)|} \int\limits_{B(x,r)} f(y) dy.$$

where $B(x,r)$ is the ball in $\mathbb{R}^n$ of radius r entered at x, and $|B(x,r)|$ is v the volume of the $B(x,r)$. By $VMO(\mathbb{R}^n)$ we denote the BMO-closure $C_0^\infty(\mathbb{R}^n)$, where $C_0^\infty(\mathbb{R}^n)$ the set of all functions from $C^\infty(\mathbb{R}^n)$ with compact support.

Let $0 < p, \theta \le \infty$, and let w be a nonnegative measurable function on $(0,\infty)$. We denote by $LM_{p\theta}^{w(\cdot)} \equiv LM_{p\theta}^{w(\cdot)}(\mathbb{R}^n)$ the general local Morrey-type space, as the space of all functions $f \in L_p^{loc}(R^n)$ with finite quasi-norm

$$\|f\|_{LM_{p\theta}^{w(\cdot)}} = \left\| w(r) \, \|f\|_{L_p(B(0,r))} \right\|_{L_\theta(0,\infty)}.$$

We denote by Ω_θ the set of all functions that are nonnegative, measurable on $(0,\infty)$, not equivalent 0 and such, that for some $t > 0$

$$\|w(r)\|_{L_\theta(t,\infty)} < \infty.$$

We denote by $\Omega_{p\theta}$ the set of all functions that are nonnegative, measurable on $(0,\infty)$, not equivalent 0 and such, that for some $t > 0$

$$\|w(r)\|_{L_\theta(t,\infty)} < \infty, \quad \|w(r) r^{\frac{n}{p}}\|_{L_\theta(0,t)} < \infty.$$

It is known the space $LM_{p\theta}^{w(\cdot)}$ is non-trivial, that is, it consists not only of functions, equivalent to 0 on $\mathbb{R}^n$, if and only if $w \in \Omega_\theta$ [1].

Theorem 1. (see. [1]) *Let $1 < p_1 < p_2 < \infty$, $\frac{1}{p_2} = \frac{1}{p_1} - \frac{\alpha}{n}$, $0 < \theta_1 \le \theta_2 < \infty$, $w_1 \in \Omega_{\theta_1}, w_2 \in \Omega_{\theta_2}$ then the condition*

$$\left\| w_2(r) \left(\frac{r}{t+r} \right)^{\frac{n}{p_2}} \right\|_{L_{\theta_2}(0,\infty)} \lesssim \|w_1(r)\|_{L_{\theta_1}(t,\infty)}, \quad t > 0, \tag{1}$$

is necessary and sufficient for boundedness of I_α from $LM_{p_1\theta_1}^{w_1(\cdot)}(\mathbb{R}^n)$ to $LM_{p_2\theta_2}^{w_2(\cdot)}(\mathbb{R}^n)$.

The next theorem contains necessary and sufficient conditions on w_1, w_2 ensuring the boundedness of I_α from from $LM_{p_1\theta_1}^{w_1(\cdot)}$ to $LM_{p_2\theta_2}^{w_2(\cdot)}$ for same values of the parameters $\alpha, p_1, p_2, \theta_1, \theta_2$.

Theorem 2. *Let $1 < p_1 < p_2 < \infty$, $0 < \alpha < \frac{n}{p_1}$, $\frac{1}{p_2} = \frac{1}{p_1} - \frac{\alpha}{n}$, $0 < \theta_1 \le \theta_2 < \infty$, $b \in BMO(\mathbb{R}^n)$, $w_1 \in \Omega_{\theta_1}, w_2 \in \Omega_{\theta_2}$ and w_1, w_2 satisfy the conditions*

$$A_0^* = sup_{t>0} \left(\int_t^\infty \left[\left(\int_r^\infty (1 + ln\frac{s}{r}) s^{-\frac{n}{p_2}-1} ds \right)^{\theta_2} w_2(r) \right] dr \right)^{\frac{1}{\theta_2}} \cdot \left[\int_t^\infty w_1^{\theta_1}(s) ds \right]^{-\frac{1}{\theta_1}} < \infty, \tag{2}$$

$$A_1^* = sup_{t>0} \left(\int_0^t w_2(s) \cdot s^{\frac{n}{p_2}} ds \right)^{\frac{1}{\theta_2}} \cdot \left(\int_t^\infty \left[\frac{\int_r^\infty (1 + ln\frac{s}{r}) s^{-\frac{n}{p_2}-1} ds}{\int_r^\infty w_1^{\theta_1}(s) ds} \right]^{\theta_1'} w_1^{\theta_1}(r) dr \right)^{\frac{1}{\theta_1'}} < \infty. \tag{3}$$

Then the commutator $[b, I_\alpha]$ is bounded operator from $LM_{p_1\theta_1}^{w_1(\cdot)}(\mathbb{R}^n)$ to $LM_{p_2\theta_2}^{w_2(\cdot)}(\mathbb{R}^n)$.

Theorem 3. *Let $1 < p_1 \leq p_2 < \infty$, $0 < \alpha < n$ and $b \in VMO(\mathbb{R}^n)$. $1 < p_1 < \frac{n}{\alpha}$, $\frac{1}{p_2} = \frac{1}{p_1} - \frac{\alpha}{n}$, $w_1 \in \Omega_{\theta_1}, w_2 \in \Omega_{p_2\theta_2}$ and the functions w_1, w_2 satisfy the conditions (1), (2), (3). Then the commutator $[b, I_\alpha]$ is a compact operator from $LM_{p_1\theta_1}^{w_1(\cdot)}$ to $LM_{p_2\theta_2}^{w_2(\cdot)}$.*

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On some embedding theorems in the space of vector-valued functions

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Let an $R^n - n$-dimensional Euclidean space, a E Banach space, an $r = (r_1, ..., r_n) > 0 - n$-dimensional vector, and $\omega_n = (1, ..., 1)$ a unit vector. $e_n = (1, ..., n)$, $e-$ subset e_n, $r_j = \overline{r}_j + \alpha_j$, r_j- integer, $0 < \alpha_j \leq 1$, $r^e = (r_1^e, ..., r_n^e)$,

$$r_j^e = \begin{cases} r_j, & if\ j \in e \\ 0, & if\ j \in e_n - e. \end{cases}$$

e_r- is the support of vector r.

Let $k = (k_1, ..., k_n) \geq 0$ be an integer, $h = (h_1, ..., h_n)$-arbitrary vector.

$$\Omega^k(f : t) = \sup_{|h| \leq t_j} \|\Delta_i^m(h) f(x)\|_{L_p(R^n:E)}$$

multiple modulus of continuity of order k of function $f(x) \in L_p(R^n : E)$, $1 \leq p$, $\theta \leq \infty$. By definition, a vector-valued strongly measurable function $f(x)$ belongs to the space $S_{p,\theta}^{r,\chi} B(R^n : E)$, if the following conditions are satisfied:

1. $\|f(x)\|_{L_p(R^n:E)} = \Big(\int ... \int\limits_{R^n} \|f(x)\|_E^p dx \Big)^{\frac{1}{p}} < \infty$

2. For $e \subset e_r$ on R^n, there are generalized mixed derivatives of $f^{r^e}(x) \in L_p(R^n : E)$.

3. For $e \subset e_r$

$$\sum_{e^1+e^2=e} \Big\{ \prod_{j \in e^2} \delta_j^\theta \int\limits_{0 \leq t_j \leq \delta_j, j \in e^1} ... \int ... \int ... \int\limits_{\delta_j \leq t_j \leq 2, j \in e^2} \prod_{j \in e^1} t_j^{-\theta\alpha_j - 1} \prod_{j \in e^2} t_j^{\theta\alpha_j - 2} \times$$
$$\times \Big[\Omega^{2\omega^e} (f^{(r^e)}(x) : t^e)_{L_p(R^n:E)} \Big]^q \Big\}^{\frac{1}{q}} \leq M_p^{r^e} \chi(\delta^e),$$

where $\sum\limits_{e^1+e^2=e}$ means that the sum is extended to all possible subsets $e^1, e^2 \subset e \subset e_r$ for which $e^1 + e^2 = e$ and $e^1 \bigcap e^2 = \emptyset$, $\chi(\delta^e) = \prod\limits_{j \in e} \chi_j(\delta_j)$, where $\chi_j(\delta_j)$ is a continuous, non-negative, non-increasing function and $\chi_j(\delta_j) = o(\delta_j)$ for $\delta_j \to 0$.

Let's define the norm

$$\|f\|_{S_{p,\theta}^r B(R^n:E)} = \sup \sum_{e \subset e_r} M_{p,\theta}^{r^e}.$$

The following embedding theorems are proved in this work.

Theorem 1. *Let* $\rho = (\rho_1, ..., \rho_n)$, $r = (r_1, ..., r_n)$, $0 < \rho \le r$, $1 \le p, \theta \le \infty$, *then the embedding*

$$S^{(r),\chi}_{p,\theta}B(R^n : E) \subset S^{(\rho),\chi}_{p,\theta}B(R^n : E)$$

and

$$\|f\|_{S^{(\rho),\chi}_{p,\theta}B(R^n:E)} \le c\|f\|_{S^{(r),\chi}_{p,\theta}B(R^n:E)}$$

is true.

Theorem 2. *Let* $e_r = e_n$, $1 \le p, \theta \le \infty$, $1 \le p \le p' \le \infty$, $\lambda = (\lambda_1, ..., \lambda_n) > 0$ *be an integer vector and the inequality*

$$\rho = r - \lambda - \Big(\frac{1}{p} - \frac{1}{p'}\Big)\omega_n > 0,$$

holds, then if $f \in S^{(r),\chi}_{p,\theta}B(R^n : E)$, *then there exists* $f^{(\lambda)}(x) \in S^{(\rho),\chi}_{p',\theta}B(R^n : E)$ *and*

$$\|f^{(\lambda)}\|_{S^{(\rho),\chi}_{p',\theta}B(R^n:E)} \le c\|f\|_{S^{(r),\chi}_{p,\theta}B(R^n:E)}.$$

Theorem 3. *Let* $r = (r_1, ..., r_n) > 0$, $e_r = e_n = e^1 + e^2$, $e^1 \bigcap e^2 = \emptyset$, $y-$ *be a point with coordinates* x_j, $j \in e^1$, z *be a point with coordinates* x_j, $j \in e^2$, $1 - \frac{1}{p}\sum\limits_{j\in e^2}\frac{1}{r_j} > 0$, $\rho = r^{e_1}$.

If $f(x) \in S^{(r),\chi}_{p,\theta}B(R^n : E)$, *then there is a trace* $\psi(y) = f(y, z_0)$, *which belongs to the class* $S^{(r^{e^1}),\chi}_{p,\theta}B(R_{e^1}, E)$ *and*

$$\|\psi : S^{(r^{e^1}),\chi}_{p,\theta}B(R_{e^1}, E)\| \le \|f' : S^{(r),\chi}_{p,\theta}B(R^n : E)\|.$$

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About the common eigen vectors of two complete continuous operators in Hilbert space

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There are some results about of the existing of the common eigenvalues of two polynomial bounded bundles. We may indicate the works [1],[2].

Now we give the sufficient, (in the special cases the sufficient and necessary) conditions for the existing of common eigen vector of two complete continuous operators A and B in Hilbert space. Operator A may be presented in the form $A = \frac{A+A^*}{2} + i\,\frac{A-A^*}{2i}$, where $T = \frac{A+A^*}{2}$ and $S = \frac{A-A^*}{2i}$ are complete continuous self-adjoint operators in Hilbert space. By analogy, we have $A = \frac{B+B^*}{2} + i\,\frac{B-B^*}{2i}$ where $M = \frac{B+B^*}{2}$ and $N = \frac{B-B^*}{2i}$ are also complete continuous self-adjoint operators in H

Let E_t and F_s and be the family of projective operators with the following properties

1. $E_a = 0, E_b = 1$

$F_c = 0, F_d = 1$

2. $E_m E_n = E_k, k = \min\ (m, n)$

$F_p F_q = F_r$, where $r = \min\ (p, q)$

3. $E_t - E_{t-0} = P_t, F_s - F_{s-0} = R_s$

where P_t is projective operator that projects onto eigen subspace of operator T , corresponding to its eigenvalue t and R_s - is the projective operator onto eigen subspace of operator S with the eigenvalue S.

Similarly, the resolutions of identity K_t and L_k of operators M and N, correspondingly, satisfy to the properties 1. $K_c = 0, K_d = 1$

$L_c = 0, L_d = 1$

2. $K_m K_n = K_k, s = \min\ (m, n)$

$L_p L_q = L_r$, where $r = \min\ (p, q)$

3. $K_t - K_{t-0} = S_t, L_t - L_{t-0} = H_t$ where S_t is projective operator onto eigen subspace of operator K, corresponding to its eigenvalue t and H_t- is projective operator onto eigen subspace of operator L, corresponding to its eigenvalue t.

Theorem 1. *If* $R(P_aR_b) \neq \{0\}$ *and* $R(S_aH_b) \neq \{0\}$ *then the complete continuous operators* A *and* B *have a common eigenvalue* $a+ib$.

Theorem 2. *Let for some four real numbers* a, b, c, d *the projective operator* $P_aR_bS_cH_d \neq 0$ *projects onto nonzero subspace then complete continuous operators* A *and* B *have a common eigen vector.*

Really, projective operator $P_aR_bS_cH_d$ projects onto intersection of eigen subspaces of operators A and B .

We have

$$P_aR_bS_cH_d \subset P_aR_b$$

$$P_aR_bS_cH_d \subset S_cH_d$$

$$P_aR_bS_cH_d \neq 0$$

Condition $R(P_aR_b) \neq \{0\}$ means that $a+ib$ is the eigenvalue of operator A.

The condition $R(S_cH_d) \neq \{0\}$ means $c+ib$ is the eigenvalue of operator B.

Condition $R(P_aR_bS_cH_d) \neq \{0\}$ means that the operators A and B have common eigen vector, so projection of projective operator $P_aR_bS_cH_d$ is contained in projections of operators P_aR_b and S_cH_d. Each element from $R(P_aR_bS_cH_d)$ is common eigen-vector of operators A and B.

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Eigenvalues of a completely continuous operators in Hilbert space

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Let A be a completely continuous operator acting in Hilbert space. The main goal of this article is to prove the sufficient condition when given complex number is eigenvalue of completely continuous operator.

It is known that the completely continuous operator A may be presented in the form

$$A = \frac{A + A^*}{2} + i\,\frac{A - A^*}{2i} \tag{1}$$

where $T = \frac{A+A^*}{2}$ and $S = \frac{A-A^*}{2i}$ are completely continuous self-adjoint operators with a countable set of eigenvalues. Besides eigenvectors corresponding to different eigenvalues of operators T and S are orthogonal, correspondingly. The principle of minimax [1] allows to determine the sequences of eigenvalues of each operator T and S.

We introduce the resolution of identity E_t of operator T and the resolution of identity F_s of operator S [2].

We have

$$1. E_a = 0, E_b = 1$$

$$F_c = 0,\ F_d = 1$$

$$2. E_m E_n = E_k,\ k = \min\ (m, n) \tag{2}$$

$$F_r = F_p F_q,\ r = \min\ (p, q)$$

$$3. E_t - E_{t-0} = P_t, F_s - F_{s-0} = R_s$$

where P_t is projective operator. If $R(P_tR_s)$ is nonzero then P_tR_s projects onto eigen subspace corresponding to eigenvalue t of operator T, and $R_s \neq 0$ is the projective operator that projects onto eigen subspace corresponding to the eigenvalue s of operator S .

Theorem. *If for some two real numbers* (t, s) *the range of two parameter projective operator* P_tR_s *is not equal to zero then* $t+is$ *is the eigenvalue of completely continuous operator* A*. If operator* A *is normal completely continuous then the condition* $P_tR_s \neq 0$ *is also necessary.*

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Spectral properties of completely continuous operators in separable Hilbert space

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Let A be a completely continuous operator in separable Hilbert space H. We represent operator A in the form $A = \frac{A+A^*}{2} + i\frac{A-A^*}{2i}$, where $T = \frac{A+A^*}{2}$ and $S = \frac{A-A^*}{2i}$ are self-adjoint completely continuous operators acting in H.

Each of them has a countable set of eigenvalues and eigenvectors corresponding to different eigenvalues are orthogonal. We denote by E_t the resolution of the identity of the operator T and by F_s the resolution of the identity of the operator S. They have the following properties

1. $E_a = 0 \quad E_b = 1$
   $F_c = 0 \quad F_d = 1$

2. $E_m E_n = E_k \quad k = \min(m, n)$
   $F_p F_q = F_r \quad r = \min(p, q)$
3. $E_t - E_{t-0} = P_t \quad F_s - F_{s-0} = R_s$,

where P_t is the projection operator that projects onto eigen subspace of operator T, corresponding to its eigenvalue t and R_s is the projection operator that projects onto eigen subspace of operator S, corresponding to its eigenvalue s.

Theorem. *Let $a + ib$ and $c + id$ be two different eigenvalues of completely continuous operator A and $P_a R_b \neq 0$ $P_c R_d \neq 0$, then the corresponding eigenvectors of operator A are orthogonal.*

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On some problems of operator theory in model spaces

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For a scalar inner function θ the model space of Sz.-Nagy and Foias is defined by

$$K_\theta := H^2 \ominus \Theta H^2,$$

where $H^2 = H^2(\mathbb{D})$ is the Hardy space over unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1$. We consider operators on the model space K_θ and prove some new results related with compactness, inevitability, invariant subspaces and other properties. We also give some applications of model technique

Solving not-self-adjoit mixed problems with separable variables

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At the beginning of the XIX century. Fourier proposed the method of separation of variables for integrating some linear partial differential equations with separable variables

$$L_1\left(\frac{\partial}{\partial t}\right)u = L_2\left(\frac{\partial}{\partial x}\right)u, \tag{1}$$

under the given boundary and initial conditions. By the Fourier method, the particular solution to the equation (1) is sought in the form $u(x,t) = T(t)X(x)$, and to determine $X(x)$ an appropriate spectral problem dependent on some parameter λ is obtained.

Let $\Delta(\lambda)$ be the denominator of the Green function [1] of this spectral problem. Denote by m the greatest multiplicity of iteration of the roots of the equation $\Delta(\lambda) = 0$.

Statement 1. If $m \geq 2$, in this case, the Fourier method is not suitable for solving the mixed problem under consideration.

For $m \geq 2$ by the particular solution of the equation (1) is sought in the form [1]

$$u(x,t) = e^{\lambda t}\left[Z_0(x) + tZ_1(x) + ... + t^{m-1}Z_{m-1}(x)\right], \tag{2}$$

where λ is a parameter, $Z_k(x)$ $(k = 0, 1, ..., m-1)$ are some functions satisfying boundary conditions.

For ease of notation and reasonings, as well as for accessibility to a wide range of readers, we explain what has been said on the following model problem.

Problem statement: to find the classic solution to the heat conductivity equation

$$\frac{\partial u}{\partial t} = a^2\frac{\partial^2 u}{\partial x^2}, \quad 0 < x < 2;\ t > 0, \tag{3}$$

Satisfying the boundary conditions

$$U_1(u) \equiv u|_{x=2} = 0, \quad U_2(u) \equiv \left.\frac{\partial u}{\partial x}\right|_{x=0} - \left.\frac{\partial u}{\partial x}\right|_{x=2} = 0; \quad (t > 0), \tag{4}$$

and the initial condition

$$u(x,t)|_{t=0} = f(x), \quad 0 < x < 2, \tag{5}$$

where $a(a > 0)-$is some number, $f(x)-$ is a specified real function .

Solution. By direct verification it is easy to be convinced that in this case, the number of iteration multiplicities of the roots of the equation will be 2, i.e. $m = 2$. By the according to formula (2) the particular solution of equation (3) will be sought in the form

$$u(x,t) = \exp(-\lambda^2 t)\left[Z_0(x) + tZ_1(x)\right], \tag{6}$$

where $Z_0(x)$ and $Z_1(x)$ are some functions satisfying the boundary conditions (4), i.e.

$$U_1(Z_k) = 0, \quad U_2(Z_k) = 0,\ k = 0, 1. \tag{7}$$

Substituting (6) in (3), we obtain

$$a^2 Z_1''(x) + \lambda^2 Z_1(x) = 0, \quad a^2 Z_0''(x) + \lambda^2 Z_0(x) = Z_1(x). \tag{8}$$

Definition 1. The boundary value problem (8)-(7) is said to be a spectral problem in the sense.

Solving boundary value problems (8)-(7) we find $Z_0 = Z_{0k}$ and $Z_1 = Z_{1k}$ and substituting these functions in (6), we have

$$u_0(x,t) = \frac{1}{2}B_0(2-x),\ u_k(x,t) = \exp(-a^2k^2\pi^2 t)\left\{[A_k \sin k\pi x + \right.$$

$$\left. + B_k(2-x)\cos k\pi x] + t2a^2k\pi B_k \sin k\pi x\right\}, \quad as\ k \geq 1, \tag{9}$$

where A_k, B_k are arbitrary numbers.

Thus, each function $u = u_k(x,t), (k = 0, 1, 2, ...)$ satisfies the equation (3) and boundary condition (4). Now we will look for the solution of problem (3)-(5) in the form

$$u(x,t) = \sum_{k=0}^{\infty} u_k(x,t). \tag{10}$$

Substituting (10) in (5), we have

$$f(x) = (2-x)\left\{\frac{1}{2}B_0 + \sum_{k=1}^{\infty} B_k \cos k\pi x\right\} + \sum_{k=1}^{\infty} A_k \sin k\pi x. \tag{11}$$

From (11) we determine unknown coefficients

$$B_k = \int_0^2 f(\xi)\cos k\pi\xi d\xi, \quad k = 0, 1, 2, ...,$$

$$A_k = \int_0^2 f(\xi)\sin k\pi\xi d\xi - \frac{1}{k\pi}B_0 - \frac{1}{2k\pi}B_k - \frac{1}{\pi}\sum_{\substack{s=1\\ s\neq k}}^{\infty} \frac{2k}{k^2-s^2}B_s. \tag{12}$$

1^0. Let a be some positive number.

2^0. Let $f(x) \in C^1([0,2])$ and $f(x)$ satisfy the boundary conditions (4), and $f'(x)$ be piecewise absolutely continuous in the interval - [0,2].

By the method used in [1], we prove the following theorem

Theorem 1. *Under the constraints 2^0 and $0 < x < 2$ we have the expansion formula (11).*

By direct verification, we prove the following theorem

Theorem 2. *Under the constraints 1^0 and 2^0, the function $u(x,t)$, determined by the formula (10) is a classic solution of the mixed problem (3)-(5).*

Note that for equations (3) the boundary conditions (4) are "well-posed" in the sense [1] and problem (3)-(5) is included in the range of problems considered in [1]. In [1] the following theorem is proved by of the finite integral transformation.

Theorem 3. *Under constraints $Rea^2 > 0$, if the problem (3)-(5) has a classic solution, then i) this solution is unique ii) and is represented by the integral formula (4) (from [1], p. 90)*

From Theorems 1-3 we obtain the following theorem

Theorem 4. *Under constraints 1^0 and 2^0 the mixed problem (3)-(5) has a unique classic solution represented by formula (10).*

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Optimality and duality analysis for non-differentiable interval valued multiobjective optimization with fuzzy environment

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This article establishes the fuzzy environment for duality analysis of non-differentiable interval-valued multiobjective optimization problems. We addressed the concept of a weak Pareto optimal solution in the fuzzy sense for an interval optimization problem in which both objective and constraint functions are non-differentiable. Moreover, the weak and strong duality relations between the primal and their dual optimization problems are established under the fuzzy pseudo/quasi-convexity assumption. Examples and applications from our study are also provided for validation.

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On the solution of an equation with a spectral parameter in the discontinuity condition

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In the paper, we consider the following problem:

$$y'' + q(x)y = \lambda^2 y, \quad 0 < x < \pi$$

$$y(a+0) = y(a-0),$$

$$y'(a-0) - y')a-0) + 2\lambda^2 m y(a) = 0,$$

$$y(0) = y(\pi) = 0.$$

Here $q(x) \in L_2(0,\pi)$ is a real-valued function, $m > 0$, $a \in (0,\pi)$ is a fixed point, λ is a complex parameter. This problem appears during the vibration of a string with fixed ends, with the mass m located at the point $x = a$ ([1]). Description of the solution $e(o,\lambda) = 1$, $e'(o,\lambda) = i\lambda$ of problem (1)-(3) satisfying the condition $e(x,\lambda)$ plays an important role in studying direct and inverse spectral problems of problem (1)-(4). This representation is in the form:

$e(x\lambda) = e^{i\lambda x} + \int_{-x}^{x} K(x,t)e^{i\lambda x}dt,\ 0 < x < a$ if,

$$e(x,\lambda) = e_0(x,\lambda) + M(x)e^{i\lambda x} + N(x)e^{i\lambda(2a-x)} + \int_{-x}^{x} K(x,t)e^{i\lambda t}dt, \quad if \quad a < x < \pi$$

here the function $e_0(x,\lambda)$ is the solution of problem (1)-(3) for $q(x) = 0$

$$M(x) = \frac{m}{2}\int_0^x q(t)dt, \quad N(x) = -\frac{m}{2}\int_0^a q(t)dt + \frac{m}{2}\int_a^x q(t)dt$$

$$K(x,-x) = \frac{\frac{m}{4}q(+0)}{1 - \frac{m}{2}\int_0^a q(t)dt}$$

$$2K'(x,x) = q(x) - \frac{m}{2}q'(x) + M(x)q(x)$$

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B-Riesz potential in the local complementary generalized B-Morrey spaces

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Let $\mathbb{R}^n_{k,+}$ be the part of the Euclidean space $\mathbb{R}^n$ of points $x = (x_1, ..., x_n)$ defined by the inequalities $x_1 > 0, \ldots, x_k > 0$, $1 \le k \le n$, $(x')^\gamma = x_1^{\gamma_1} \cdot \ldots \cdot x_k^{\gamma_k}$, $\gamma = (\gamma_1, ..., \gamma_k)$ is a multi-index consisting of fixed positive numbers.

In this paper we realize some estimations of the B-Riesz potential generated by the generalized shift operator of the form

$$T^y f(x) = C_{\gamma,k} \int_0^\pi \ldots \int_0^\pi f\left((x', y')_\beta, x'' - y''\right)\, d\nu(\beta),$$

where $(x_i, y_i)_{\beta_i} = (x_i^2 - 2x_i y_i \cos\beta_i + y_i^2)^{\frac{1}{2}}$, $1 \le i \le k$, $(x', y')_\beta = ((x_1, y_1)_{\beta_1}, \ldots, (x_k, y_k)_{\beta_k})$, $d\nu\,(\beta) = \prod\limits_{i=1}^{k} \sin^{\gamma_i - 1}\beta_i\, d\beta_1 \ldots d\beta_k$, $1 \le k \le n$ and

$$C_{\gamma,k} = \pi^{-\frac{k}{2}}\, \Gamma^{-1}\left(\frac{|\gamma|}{2}\right) \prod_{i=1}^{k} \Gamma\left(\frac{\gamma_i + 1}{2}\right) = \frac{2^{k-1}|\gamma|}{\pi}\left(\frac{|\gamma|}{2} + 1\right)\omega(2, k, \gamma).$$

Consider the B-Riesz potential

$$I_{\alpha,\gamma} f(x) = \int_{\mathbb{R}^n_{k,+}} T^y[|f|\,](x)|y|^{\alpha - Q}(y')^\gamma dy, \quad 0 < \alpha < Q.$$

Let $1 \leq p < \infty$, ω positive measurable function. The local "complementary" generalized B-Morrey space ${}^{\complement}\mathcal{M}_{\{0\}}^{p,\omega,\gamma}(\mathbb{R}_{k,+}^n)$ is defined by the norm

$$\|f\|_{{}^{\complement}\mathcal{M}_{\{0\}}^{p,\omega,\gamma}(\mathbb{R}_{k,+}^n)} = \sup_{t>0} \frac{t^{\frac{Q}{p'}}}{\omega(t)} \left(\int_{\mathbb{R}_{k,+}^n \setminus E(0,t)} T^y[|f|\ ]^p(x)(y')^\gamma dy \right)^{1/p}.$$

Theorem 1. *Let $0 < \alpha < Q$, $1 < p < \frac{Q}{\alpha}$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{Q}$, the functions $\omega_1(r)$ and $\omega_2(r)$ fulfill the condition*

$$\int_0^t \omega_1(r) \frac{dr}{r} \leq C\, t^{-\alpha} \omega_2(t),$$

where C does not depend on t. Then the operator$I_{\alpha,\gamma}$ is bounded from ${}^{\complement}\mathcal{M}_{\{0\}}^{p,\omega_1,\gamma}(\mathbb{R}_{k,+}^n)$ to ${}^{\complement}\mathcal{M}_{\{0\}}^{q,\omega_2,\gamma}(\mathbb{R}_{k,+}^n)$.

Weighted inequality for fractional maximal operator in generalized Morrey spaces

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One of the most important variants of the Hardy-Littlewood maximal function is the so-called fractional maximal function defined by the formula

$$M_\alpha f(x) = \sup_{t>0} |B(x,t)|^{-1+\alpha/n} \int\limits_{B(x,t)} |f(y)| dy,\ 0 \leq \alpha < n,$$

where $|B(x,t)|$ is the Lebesque measure of the ball $B(x,t)$.

It coincides with the Hardy-Littlewood maximal function $Mf \equiv M_0 f$ and is indimately related to the Riesz potential

$$I_\alpha f(x) = \int\limits_{R^n} \frac{f(y)dy}{|x-y|^{n-\alpha}},\ 0 \leq \alpha < n.$$

Let $L_{p,\omega}(R^n)$ be the space of measurable functionson R^n with finite norm

$$\|f\|_{L_{p,\omega}} = \|f\|_{L_{p,\omega}(R^n)} = \Big(\int\limits_{R^n} |f(x)|^p \omega(x) dx \Big)^{1/p}, \quad 1 \le p < \infty$$

and for $p = \infty$ the spase $L_{\infty,\omega}(R^n) = L_\infty(R^n)$.

Definition. *Let $1 \le p < \infty$. The generalized Morrey space $M^{p,\omega}(R^n)$ and generalized weighted Morrey space $M^{p,\omega,|\cdot|^\gamma}(R^n)$ are defined by the norms*

$$\|f\|_{M^{p,\omega}} = \sup_{x \in R^n, r>0} \frac{r^{-\frac{n}{p}}}{\omega(x,r)} \|f\|_{L_p(B(x,r))},$$

$$\|f\|_{M^{p,\omega,|\cdot|^\gamma}} = \sup_{x \in R^n, r>0} \frac{r^{-\frac{n}{p}}}{\omega(x,r)} \|f\|_{L_{p,|\cdot|^\gamma}(B(x,r))}.$$

Theorem 1. *Let $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}$, $\alpha p - n < \gamma < n(p-1)$, $\mu = \frac{q\gamma}{p}$ and the functions $\omega_1(x,r)$ and $\omega_2(x,r)$ fulfill the condition*

$$\int\limits_r^\infty t^{\alpha - \frac{\gamma}{p}} \omega_1(x,t) \frac{dt}{t} \le C r^{-\frac{\gamma}{p}} \omega_2(x,r).$$

Then the operators M^α and I^α are bounded from $M^{p,\omega_1(\cdot),|\cdot|^\gamma}(R^n)$ to $M^{q,\omega_2(\cdot),|\cdot|^\gamma}(R^n)$.

Theorem 2. *Let $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}$, γ, μ, satisfy condition*

$$0 \le \gamma < \frac{n}{p'}, \quad \mu = \frac{\gamma}{p},$$

and let $\omega(x,t)$ satisfy condition

$$r^\alpha \omega(x,r) \le C,$$

Then the operator M^α is bounded from $M^{p,\omega(\cdot),|\cdot|^\gamma}(R^n)$ to $L_{\infty,|\cdot|^\mu}(R^n)$.

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On absolute and uniform convergence of spectral expansion in eigen-functions of Dirac operator

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Consider on the interval $G = (0, \pi)$ one-dimensional Dirac operator

$$Du = B\frac{du}{dx} + P(x)u, \quad u(x) = \left(u^1(x),\, u^2(x)\right)^{\,T},$$

where $B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $P(x) = diag\,(p(x),\, q(x))$, moreover $p(x)$ and $q(x)$ are real functions belonging to $L_2\,(0, \pi)$.

Let $L_p^2(G)$, $p \geq 1$, be a space of two-dimensional vector-functions $f(x) = (f_1(x),\, f_2(x))^{\,T}$ with the norm

$$\|f\|_{p,2} = \left(\int_G |f(x)|^p\,dx\right)^{1/p}, \quad \left(\|f\|_{\infty,2} = \sup_G vrai\,|f(x)|\right), \quad .$$
$$|f(x)| = \left(|f_1(x)|^2 + |f_2(x)|^2\right)^{1/2}$$

Obviously, for $f(x) \in L_p^2(G)$, $g(x) \in L_q^2(G)$, $p^{-1} + q^{-1} = 1$, $p \geq 1$, there exists the “scalar productl”

$$(f, g) = \int_G < f,\ g > dx = \int_G \sum_{j=1}^{2} f_j(x)\overline{g_j(x)}dx\,.$$

Following [1], under the vector-function of the operator D, responding to the real eigen-value λ, we will understand any identically non-zero vector-function $y(x) = (y^1(x), y^2(x))^T$, which is absolutely continuous on $\bar{G} = [0, \pi]$ and almost everywhere in G satisfies the equation $Dy = \lambda y$.

Let $\{u_n(x)\}_{n=1}^{\infty}$ be a complete orthonormalized in $L_2^2(G)$ system consisting of eigen vector-functions of the operator D, and $\{\lambda_n\}_{n=1}^{\infty}, \lambda_n \in R$, be an appropriate system of eigen values.

For the vector-functions $f(x) \in W_p^1(G)$, $p \geq 1$, we introduce a partial sum of its spectral expansion in the system $\{u_n(x)\}_{n=1}^{\infty}$:

$$\sigma_v(x, f) = \sum_{|\lambda_n| \leq v} f_n u_n(x) = \sum_{|\lambda_n| \leq v} (f, u_n)\, u_n(x), \quad v \geq 1.$$

Denote $R_v(x, f) = \sigma_v(x, f) - f(x)$ and

$$A_n(f) = \langle f, Bu_n \rangle \Big|_0^{\pi} = \left(f_1(x)\overline{u_n^2(x)} - f_2(x)\,\overline{u_n^1(x)} \right) |_0^{\pi},$$

$$f(x) = (f_1(x), f_2(x))^T, \quad u_n(x) = \left(u_n^1(x),\, u_n^2(x)\right)^T.$$

The following theorems are the main results of this paper.

Theorem. *Let $p(x),\ q(x) \in L_r(G),\ r > 1,$ and the vector-function $f(x)$ belong to $W_p^1(G),\ 1 < p \leq \infty$. Then*

a) *for the uniform convergence of the series*

$$\sum_{n=1}^{\infty} |f_n|\ |u_n(x)|\ , \quad x \in \bar{G} \tag{1}$$

it is necessary and sufficient uniform convergence of the series

$$\sum_{|\lambda_n| \geq 1} |\lambda_n|^{-1} |A_n(f)|\ |u_n(x)|\ , \ \ x \in \bar{G}; \tag{2}$$

b) *for the uniform convergence on $\bar{G}$ of the spectral expansion*

$$\sum_{n=1}^{\infty} f_n u_n(x) \tag{3}$$

it is necessary and sufficient uniform convergence of the series

$$\sum_{|\lambda_n|\geq 1} \lambda_n^{-1} A_n(f)\, u_n(x)\,, \quad x \in \bar{G}. \tag{4}$$

c) if $A_n(f) = 0,\ n = 1, 2, ...,$then spectral expansion (3) of the vector-function $f(x)$ converges absolutely and uniformly on $\bar{G} = [0, \pi]$ and the following estimations are valid

$$\|R_v(\cdot, f)\|_{C[0,\pi]} \leq const\, v^{-\delta} \left\{ \|Pf\|_{2,2} + \|f\|_{W_p^1(G)} \right\}, \tag{5}$$

$$\|R_v(\cdot, f)\|_{C[0,\pi]} = o\left(v^{-\delta}\right), \quad v \to +\infty, \tag{6}$$

where $\delta = \min\{1/q,\ 1/2\}$, $p^{-1} + q^{-1} = 1$, *const is independent of* f, *the symbol "o" is dependent on* f.

Remark. If the system $\{u_n(x)\}_{n=1}^{\infty}$ and the vector-function $f(x) = (f_1(x), f_2(x))^T$ satisfy one of the self-adjoint conditions

$$u^1(0) = u^2(\pi) = 0\,;$$

$$u^1(0) + \omega u^1(\pi) = 0, \ \ \bar{\omega} u^2(0) + \beta_1 u^1(\pi) + u^2(\pi) = 0\,;$$

$$\beta_2 u^1(0) + u^2(0) + \omega u^1(\pi) = 0\,, \ \ -\omega u^1(0) + \beta_3 u^1(\pi) + u^2(\pi) = 0$$

then $A_n(f) = 0,\ n = 1, 2, \ldots$. Where $\beta_i,\ i = \overline{1,3}$ are arbitrary real numbers, $\omega \neq 0$ is an arbitrary complex number,

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Variational principle for a two-parameter spectral problem using a linear functional

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Consider the two-parameter problem

$$\lambda_1 K_{i,1}\varphi_i + \lambda_2 K_{i,2}\varphi_i = \varphi_i,\ \varphi_i \in\ H_i\, i = 1; 2 \tag{1}$$

with compact self-adjoint operators $K_{i,1}$, $K_{i,2}$, $i = 1;\ 2$ in Hilbert space H_i, $i = 1;\ 2$. Consider also the problem

$$\lambda_1 K^t_{i,1}\varphi + \lambda_2 K^t_{i,2}\varphi = \varphi,\ \varphi \in\ H = H_1 \otimes H_2,\ i = 1;\ 2 \tag{2}$$

and the problem

$$\lambda_i \Delta_0\varphi = \Delta_i\varphi, \quad \varphi \in\ H = H_1 \otimes H_2,\ i = 1;\ 2, 8 \tag{3}$$

where $K^t_{1,i,} = K_{1,i} \otimes I_2$ and $K^t_{2,i} = I_1 \otimes K_{2,i}$, $i = 1;\ 2$ are operators in the Hilbert space $H = H_1 \otimes H_2$ and $I_1,\ I_2$ unit operators in the spaces $H_1,\ H_2$ respectively, $\Delta_0 = K_{1,1} \otimes K_{2,2} - K_{1,2} \otimes K_{2,1}$, $\Delta_1 = I_1 \otimes K_{2,2} - K_{1,2} \otimes I_2$, $\Delta_2 = K_{1,1} \otimes I_2 - I_1 \otimes K_{2,1}$.

Let the foloving conditions be satisfied

$$\Delta_1 > 0, \Delta_2 > 0, \ker \Delta_0 = \{0\}, \tag{4}$$

We will call the set

$$M = \left\{ (\lambda_1,\ \lambda_2\ )/\,(\lambda_1,\ \lambda_2\ ) = \left(\frac{(\Delta_1\,\varphi,\varphi)}{(\Delta_0\,\varphi,\varphi)}, \frac{(\Delta_2\,\varphi,\varphi)}{(\Delta_0\,\varphi,\varphi)} \right) \quad \forall\varphi \in \mathrm{H} = H_1 \otimes H_2 \right\}$$

the numerical range of problem (1). The numerical range of a multiparameter problem in various forms is studied in [1]-[5]. Here, the variational principle is studied for finding the eigenelements and corresponding eigenvalues of problem (1) using a linear functional. The condition

$$\Delta_1 > 0, \Delta_2 > 0, \Delta_2 > 0 \tag{5}$$

is called a complete definiteness condition.

Theorem 1. *Under condition (5), there is always a decomposable tensor* $\exists\varphi^0 = \varphi_1^0 \otimes \varphi_2^0 \in H'$ *such that the functional* $F(\varphi,\varepsilon) = \frac{(\Delta_1\,\varphi,\varphi)+(1+\varepsilon)(\Delta_2\,\varphi,\varphi)}{(\Delta_0\,\varphi,\varphi)}$ *reaches the minimum value for any sufficiently small value* $\varepsilon > 0$ *and the element* φ^0 *is an eigenelement of problem (1) corresponding to the eigenvalue* $(\lambda_1^0,\, \lambda_2^0) = \left(\frac{(\Delta_1\,\varphi^0,\varphi^0)}{(\Delta_0\,\varphi^0,\varphi^0)}, \frac{(\Delta_2\,\varphi^0,\varphi^0)}{(\Delta_0\,\varphi^0,\varphi^0)}\right)$. *Here the set* $H' \subset H = H_1 \otimes H_2$ *is the set of decomposable tensors.*

Theorem 2. *Under the definiteness condition (5), there exists a sequence of decomposable tensors* $\{\varphi^n\} = \{\varphi_{1n} \otimes \varphi_{2n}\}$, $\varphi_{1n} \in H_1,\ \varphi_{2n} \in H_2,\ \varphi^n \in H'$, *for which all elements of this sequence are eigenelements of problem (1) and this sequence of eigenelements constitutes a complete orthonormal basis in the space* $H_{\Delta_0} = H_1 \otimes\ H_2$, *i.e.* $[\varphi^n,\ \varphi^m] \equiv (\Delta_0\varphi^n,\ \varphi^m) = \begin{cases} 0, & if\ n \neq m \\ 1, & if\ n = m \end{cases}$.

If for problem (1) the definiteness condition is satisfied not in the form of (5) but in the form of (4), then a similar theorem can be proven and a sequence of eigenelements and corresponding eigenvalues of problem (1) can be constructed.

Theorem 3. *Under the definiteness condition (4), there exists a sequence of decomposable tensors* $\{\varphi^n\} = \{\varphi_{1n} \otimes \varphi_{2n}\}$, $\varphi_{1n} \in H_1,\ \varphi_{2n} \in H_2,\ \varphi^n \in H'$, $H' \subset H_{|\Delta_0|} = H_1 \otimes H_2$, *for which all elements of this sequence are eigenelements of problem (1) and this sequence of eigenelements constitutes a complete orthonormal basis in the space* $H_{|\Delta_0|} = H_1 \otimes\ H_2$

Depending on the application of the variational principle, it is sometimes useful to formulate Theorem 1 in the following form.

Theorem 4. *Under condition (4), for any pair of real numbers* $a > 0$ *and* $b > 0$ *there always exists such a decomposable tensor* $\exists\varphi^0 = \varphi_1^0 \otimes \varphi_2^0 \in H = H_1 \otimes H_2$ *that the functional* $F_1(\varphi, a,\, b,\, \varepsilon) = \frac{a\,(\Delta_1\,\varphi,\varphi)+(b+\varepsilon)(\Delta_2\,\varphi,\varphi)}{|(\Delta_0\,\varphi,\varphi)|}$ *(or* $F_2(\varphi, a,\, b,\, \varepsilon) = \frac{(a+\varepsilon)\,(\Delta_1\,\varphi,\varphi)+b\,(\Delta_2\,\varphi,\varphi)}{|(\Delta_0\,\varphi,\varphi)|}$*) reaches the minimum value for any sufficiently small value* $\varepsilon > 0$ *and the element* φ^0 *is an eigenelement of problem (1) corresponding to the eigenvalue* $(\lambda_1^0,\, \lambda_2^0) = \left(\frac{(\Delta_1\,\varphi^0,\varphi^0)}{(\Delta_0\,\varphi^0,\varphi^0)}, \frac{(\Delta_2\,\varphi^0,\varphi^0)}{(\Delta_0\,\varphi^0,\varphi^0)}\right)$.

Theorem 5. *Under condition (4), in the first quadrant there exists a cone* $S_1 \subset R^2$ *with a vertex at the point* $(0,\, 0)$ *and in the third quadrant a cone* $S_2 \subset R^2$ *with a vertex at the point* $(0,\, 0)$, *the union of which contains the* *numerical range of problem (1).*

The proof of this theorem is easily obtained from the definition of a numerical range and of the left definiteness condition (4). Since, under the condition (4), the numerical range of problem (1) is in the union of the first and third quadrants. And each quadrant is a cone with a vertex at the point (0, 0). But generally speaking, these cones can have an acute angle at the vertex. Here we will introduce a technique for accurately determining this angle.

Theorem 6. *If under condition (4), the functional* $F_1(\varphi) = \frac{(\Delta_1\varphi,\varphi)(\Delta_2\varphi,\varphi)}{(\Delta_0\varphi,\varphi)^2}$ *reaches its minimum value in an element* $\varphi^0 = \varphi_1^0 \otimes \varphi_2^0 \in H = H_1 \otimes H_2$, *then there exists a pair of positive real numbers* (a, b) *such that at least one of the functionals* $F_2(\varphi, a, b, \varepsilon) = \frac{a\,(\Delta_1\varphi,\varphi)+(b+\varepsilon)\,(\Delta_2\varphi,\varphi)}{|(\Delta_0\varphi,\varphi)|}$ *or* $F_3(\varphi, a, b, \varepsilon) = \frac{(a+\varepsilon)\,(\Delta_1\varphi,\varphi)+b\,(\Delta_2\varphi,\varphi)}{|(\Delta_0\varphi,\varphi)|}$ *reaches the minimum value in this element for any sufficiently small positive value* $\varepsilon > 0$.

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On the solvability of the inverse spectral problem for Sturm-Liouville operator with a spectral parameter square including the boundary condition

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Inverse problems of spectral analysis consist in determining operators from their known spectral data, which can include one, two or more spectra, a spectral function, a spectrum and normalization numbers, a Weyl function, etc. Depending on the choice of spectral data, various formulations of inverse problems are possible.

In this paper, we study the inverse spectral problem of reconstructing the Sturm-Liouville operator with nonseparated boundary conditions, one of which contains a quadratic function of the spectral parameter. Sufficient conditions for the solvability of the inverse problem are obtained. The spectrum of one boundary value problem, a certain sequence of signs, and a certain number are used as spectral data. Note that in works [1-2] a similar problem was completely solved in the case of other boundary conditions. A review of results related to solutions of inverse problems for differential operators with nonseparated boundary conditions is available in [2-5].

Let us consider the boundary value problem generated on the interval $[0, \pi]$ by the Sturm-Liouville equation

$$-y'' + q(x)\, y = \lambda^2 y \tag{1}$$

and boundary conditions of the form

$$\begin{gathered} y(0) - y(\pi) = 0, \\ y'(0) - \left(m\lambda^2 + \alpha\lambda + \beta\right) y(\pi) - y'(\pi) = 0, \end{gathered}$$

where $q(x)$ is a real function belonging to the space $L_2[0,\pi]$, λ is a spectral parameter, α, β, m are real numbers and $m\alpha \neq 0$. We will denote this problem by P.

Let $c(x,\lambda)$, $s(x,\lambda)$ the fundamental system of solutions to equation (1), determined by the initial conditions

$$c(0,\lambda) = s'(0,\lambda) = 1, c'(0,\lambda) = s(0,\lambda) = 0.$$

The spectrum of the problem P coincides with the set of zeros of an entire function of exponential type

$$\Delta(\lambda) = c(\pi,\lambda) + \left(m\lambda^2 + \alpha\lambda + \beta\right) s(\pi,\lambda) + s'(\pi,\lambda) - 2,$$

which is called the characteristic function of the boundary value problem P. For the eigenvalues of the μ_k $(k = \pm 0, \pm 1, \pm 2, ...)$ as $|k| \to \infty$ of the following asymptotic formula holds:

$$\mu_k = k + \frac{2\left[(-1)^k - 1\right] + mA}{\pi\, m\, k} + \frac{\tau_k}{k}, \{\tau_k\} \in l_2. \tag{2}$$

Let us denote $\sigma_n = \operatorname{sign}[1 - |\, s'(\pi, \lambda_n)|]$, $n = \pm 1, \pm 2, ...,$ where λ_n are the zeros of the function $s(\pi,\lambda)$, whose squares are the eigenvalues of the boundary value problem generated by equation (1) and Dirichlet boundary conditions $y(0) = y(\pi) = 0$. Sequences $\{\mu_k\}$, $\{\sigma_n\}$ and number β will be called spectral data of the boundary value problem P.

Theorem. *In order for sequences of real numbers $\{\mu_k\}$, $\{\sigma_n\}$ $(\sigma_n = -1, 0, 1\ ;\ n = \pm 1, \pm 2, ...)$, and a real number β were spectral data of a boundary value problem of the form P, it is sufficient for the following conditions to be satisfied:*

1) *there is an asymptotic formula (2), in which $m \neq 0$, $A-$ real numbers;*

2) *the terms of the given sequence $\{\mu_k\}$ and the sequence $\{\lambda_k\}$ of zeros of the function $\Delta(\lambda) - \Delta(-\lambda)$ are interleaved in the sense*

$$... \leq \lambda_{-3} \leq \mu_{-2} \leq \lambda_{-2} \leq \mu_{-1} \leq \lambda_{-1} \leq \mu_{-0} < 0 < \mu_{+0} \leq$$

$$\leq \lambda_1 \leq \mu_1 \leq \lambda_2 \leq \mu_2 \leq \lambda_3 \leq ... ,$$

where

$$\Delta(\lambda) = m\pi(\mu_{-0} - \lambda)(\mu_{+0} - \lambda) \prod_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{\mu_k - \lambda}{k} \text{ and } \lambda_i \neq \lambda_j \text{ at } i \neq j;$$

3) *the inequality* $|c_n| \geq 2$ *is true, where* $c_n = \Delta(\lambda_n) + 2$*;*

4) σ_n *takes a value equal to zero if* $|c_n| = 2$*, and a value of 1 or -1 if* $|c_n| > 2$*, there is* $N > 0$*, such that* $\sigma_n = 1$ *for all* $|n| \geq N$*.*

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Optimization of second order delay-differential inclusions

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The present paper studies a new class of problems of optimal control theory with delay-differential inclusions (DFIs). Discrete and continuous time processes with first order ordinary discrete-differential and partial differential inclusions found wide application in the field of mathematical economics and in problems of control dynamic system optimization and differential games (see [2]-[14] and their references). Observe that such problems arise frequently not only in mechanics, aerospace engineering, management sciences, and economics, but also in problems of automatic control, aviovibration, and biophysics [4]. Optimization of higher order differential inclusions was first developed by Mahmudov in [4, 7, 10, 11]. Here we recall the key notions of set-valued mappings from the book [4]; let $\mathbb{R}^n$ be a n-dimensional Euclidean space, $\langle x, v\rangle$ be an inner product of elements $x, v \in \mathbb{R}^n$, (x, v) be a pair of x, v.

The LAM to nonconvex mapping F is defined as follows

$$
\begin{aligned}
&F^*\big(v^*;(x,u_1,u_2,v)\big) = \big\{(x^*,u_1^*,u_2^*) : H_F(x^1,u_1^1,u_2^1,v^*) - (x,u_1,u_2,v^*) \\
&\geq \langle x^*, x^1 - x\rangle + \langle u_1^*, u_1^1 - u_1\rangle + \langle u_2^*, u_2^1 - u_2\rangle,\ \forall (x^1,u_1^1,u_2^1) \in \mathbb{R}^{3n}\big\}, \\
&\quad (x,u_1,u_2,v) \in gphF,\ v \in F_A(x,u_1,u_2;v^*), \text{where} F_A \text{is an argmaximum set.}
\end{aligned}
$$

For most of this paper, we consider optimization problems with second-order delay-DFIs and state constraints of the form, labeled as (PCD):

$$
\begin{aligned}
&\text{minimize } J[x(\cdot)] = \textstyle\int_0^T g(x(t),t)dt,\ x''(t) \in F(x(t),x'(t),x(t-h),t), \\
(PCD) \qquad &x(t) = \xi(t),\ t \in [-h,0),\ x(0) = \theta,\ x(T) \in P,\ \text{a.e. } t \in [0,T],
\end{aligned}
$$

where $F(\cdot,t) : \mathbb{R}^{3n} \rightrightarrows \mathbb{R}^n$ and $g(\cdot,t)$ are time dependent set-valued mapping and continuous proper function, respectively, $P \subseteq \mathbb{R}^n, \xi(t), t \in [-h,0)$ is an absolutely continuous initial function, θ is a fixed vector. It is required to find a feasible trajectory (arc) $x(t), t \in [-h,T]$ minimizing the functional $J[x(\cdot)]$

over a set of feasible trajectories. Here, a feasible trajectory $x(t), t \in [-h,T]$ almost everywhere (a.e.) the second order delay-DFI (with a possible jump discontinuity at $t = 0$), $x(\cdot) \in AC\big([-h,T]\big) \cap W^n_{1,2}\big([0,T]\big)$. At first, let us formulate an adjoint delay-DFIs for a convex problem (PCD):

$$
\begin{aligned}
&(i)\Big(x^{*''}(t) + \psi^{*'}(t) - \eta^*(t+h), \psi^*(t), \eta^*(t)\Big) \in F^*\big(x^*(t); \big(\tilde{x}(t), \tilde{x}^{'}(t), \tilde{x}(t-h),\\
&\quad \tilde{x}^{''}(t)\big), t\big) - \{\partial g\big(\tilde{x}(t), t\big)\} \times \{0\} \times \{0\}, \text{a.e.}\, t \in [0, T-h), x^*(0) = 0;\\
&(ii)\ \Big(x^{*''}(t) + \psi^{*'}(t), \psi^*(t), \eta^*(t)\Big) \in F^*\big(x^*(t); \big(\tilde{x}(t),\ \tilde{x}^{'}(t), \tilde{x}(t-h),\ \tilde{x}^{''}(t)\big), t\big)\\
&\quad - \{\partial g\big(\tilde{x}(t), t\big)\} \times \{0\} \times \{0\},\ \text{a.e.}\, t \in [T-h, T],\\
&(iii)\ - x^{*'}(T) - \psi^*(T) \in K^*_P\big(\tilde{x}(T)\big)\ x^*(T) = 0.
\end{aligned}
$$

Here we assume that $x^*(t), t \in [0,T]$, is an absolutely continuous function together with the first order derivatives for which $x^{*''}(\cdot) \in L^n_1[0,T]$. Moreover $\psi^*(t), \eta^*(t), t \in [0,T] t \in [0,T]$ are absolutely continuous and $\psi^{*'}(\cdot) \in L^n_1[0,T]$. The condition guaranteeing nonemptiness of the LAM F^* at a given point is the following $(iv)\tilde{x}''(t) \in F_A\big(\tilde{x}(t), \tilde{x}^{'}(t), \tilde{x}(t-h); x^*(t), t\big)$, a.e. $t \in [0,T]$.

Theorem 1 *Let $g : \mathbb{R}^n \times [0,T] \to \mathbb{R}^1$ be continuous and convex with respect to x function, and $F(\cdot, t) : \mathbb{R}^{3n} \rightrightarrows \mathbb{R}^n$ be a convex set-valued mapping. Then for optimality of the arc $\tilde{x}(t)$ to the convex problem (PCD) with second order delay-DFIs it is sufficient that there exist a triple $\{x^*(t), \psi^*(t), \eta^*(t)\}$ of absolutely continuous functions, $x^*(t), \psi^*(t), \eta^*(t), t \in [0,T]$, satisfying a.e. the conditions (i)-(iv).*

Let us consider an example:

$$
\begin{aligned}
&\text{minimize } J[x(\cdot)],\ x^{''}(t) = A_0 x(t) + A_1 x^{'}(t) + A_2 x(t-h) + Bu(t),\\
&(PLD) \qquad\qquad x(t) = \xi(t),\ t \in [-h, 0),\ x(0) = \theta,\ x(T) \in P,\ \text{a.e. } t \in [0,T],
\end{aligned}
$$

where g is continuously differentiable function in $x, A_i, i = 0,1,2$ and B are $n \times n$ and $n \times r$ matrices, respectively, $U \subseteq \mathbb{R}^r$ is a convex compact. The problem is of finding corresponding to the controlling parameter $\tilde{u}(t) \in U$ and arc $\tilde{x}(t)$, minimizing $J[x(\cdot)]$ over a set of feasible solutions.

Theorem 2 *The arc $\tilde{x}(t)$ corresponding to the controlling parameter $\tilde{u}(t)$ minimizes $J[x(\cdot)]$ over a set of feasible solutions in the convex second order delay-differential problem (PLD), if there exists an absolutely continuous function $x^*(t)$ together with the first order derivatives, satisfying the conditions:*

$$x^{*''}(t) = A_0^* x^*(t) - A_1^* x^{*'}(t) + A_2^* x^*(t+h) - g'(\tilde{x}(t), t),\ t \in [0, T-h);$$
$$x^{*''}(t) = A_0^* x^*(t) - A_1^* x^{*'}(t) - g'(\tilde{x}(t), t),\ t \in [T-h, T],\ x^*(0) = x^*(T) = 0,$$
$$x^{*'}(T) \in K_P^*\big(\tilde{x}(T)\big); \langle B\tilde{u}(t), x^*(t)\rangle = \sup_{u\in U}\langle Bu, x^*(t)\rangle,\ t \in [0, T].$$

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On Jensen's inequality with applications

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In this paper, we present some interesting improvements of Jensen's inequality. The results have been obtained by a new approach using Green convex functions, Lah-Ribarić inequality, and weighted Hermite-Hadamard inequality. Moreover, the conditions of the Jensen-Steffensen and converse of Jensen's inequality have been utilized for the derivation of improvements of Jensen's inequality. As applications, improvements of power means, quasi-arithmetic means, and Hermite-Hadamard inequalities have been obtained. At the end, some applications are presented in information theory.

Mean ergodic theorem for probability measures

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Let G be a locally compact group with the left Haar measure m_G and let $M(G)$ be the convolution measure algebra of G. As usual, $C_0(G)$ will denote the space of all complex-valued continuous functions on G vanishing at infinity. Since $C_0(G)^* = M(G)$, the space $M(G)$ carries the weak* topology $\sigma(M(G), C_0(G))$. In the following, the w^*-topology on $M(G)$ always means this topology. For a subset S of G, by $[S]$ we will denote the closed subgroup of G generated by S. A probability measure $\mu \in M(G)$ is said to be *adapted* if $[supp\mu] = G$. Also, a probability measure $\mu \in M(G)$ is said to be *strictly aperiodic* if the support of μ is not contained in a proper closed left cosets gH $(H \neq G, g \in G \diagdown H)$ of G.

The classical Kawada-Itô theorem [3, Theorem 7] asserts that if μ is an adapted measure on a compact metrisable group G, then the sequence of probability measures $\left\{\frac{1}{n+1}\sum\limits_{i=0}^{n}\mu^i\right\}_{n\in\mathbb{N}}$ weak* converges to the Haar measure on G.

If μ is an adapted and strictly aperiodic measure on a compact metrisable group G, then w*- $\lim\limits_{n\to\infty} \mu^n = m_G$ [3, Theorem 8]. For related results see also, [1, 2, 5, 6].

It is well known that if G is a compact group, then the (normalized) Haar measure m_G is an idempotent measure on G with $suppm_G = G$. If H is a closed subgroup of G, then the measure m_H may be regarded as a measure on G by putting $\overline{m}_H(B) = m_H(B \cap H)$ for every Borel subset B of G. Notice that $supp\overline{m}_H = H$.

The following result combines Kawada-Itô theorems.

Theorem. *Let μ and ν be two probability measures on a compact group G and assume that ν is strictly aperiodic. Then we have*

$$\mathrm{w}^* - \lim_{n\to\infty} \frac{1}{n+1} \sum_{i=0}^{n} \mu^i * \nu^{n-i} = \overline{m}_{[supp\mu]} * \overline{m}_{[supp\nu]}.$$

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Fredholmness of the dirichlet problem for $2m$-th order elliptic equations in Grand Sobolev Spaces

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In this paper, on a bounded domain $\Omega \subset R^n$ with a sufficiently smooth boundary $\partial\Omega$, it is considered a uniformly elliptic equation of $2m$ order, the coefficients of which are continuous in the principal part. Grand Lebesgue space $L_{p)}(\Omega)$, $1 < p < +\infty$, is considered. This space is non-separable and it is defined a separable subspace $N_{p)}(\Omega)$ of $L_{p)}(\Omega)$, in which infinitely differentiable functions are dense. It is defined the grand Sobolev space $N^{2m}_{p)}(\Omega)$ of $2m$-th order differentiable in Sobolev sense functions generated by the subspace $N_{q)}(\Omega)$. For this equation, a Schauder-type estimate up to the boundary is proved. Using this estimate, we establish an a priori estimate and then the Fredholmness of the $2m$-order elliptic equation under consideration in $N^{2m}_{q)}(\Omega)$. By solution, we mean a strong solution.

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A new approach to the study of $m - cv$ functions

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1. $m-$convex ($m-cv$)functions are a further continuation, development of convex functions in geometry. The class of convex functions ($m = 1$) has been well studied in the works of A. Aleksandrov [4], I. Bakelman [6], A. Pogorelov [5], A. Artykbaev [7], etc. For $m > 1$ this class was studied in a series of works by N. Ivochkina, N. Trudinger, X. Wang, S. Li, H. Lu et al (see [8]).

For a twice smooth functions $u(x) \in C^2(D)$, $D \subset \mathbb{R}^n$, they are defined by Hessians $H_k(u), 1 \leq k \leq n$. Recal, for $u(x) \in C^2(D)$ the matrix $(\frac{\partial^2 u}{\partial x_j \partial x_t})$ is orthogonal, $\frac{\partial^2 u}{\partial x_j \partial x_t} = \frac{\partial^2 u}{\partial x_t \partial x_j}$. Therefore, after a suitable orthonormal transformation, it is transformed into a diagonal form,

$$\left(\frac{\partial^2 u}{\partial x_j \partial x_t}\right) \to \begin{pmatrix} \lambda_1 & 0 & ... & 0 \\ 0 & \lambda_2 & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & \lambda_n \end{pmatrix},$$

where $\lambda_j = \lambda_j(x) \in \mathbb{R}$ are the eigenvalues of the matrix $(\frac{\partial^2 u}{\partial x_j \partial x_t})$. Let $H_k(u) = H_k(\lambda) = \sum\limits_{1 \leq j_1 < ... < j_k \leq n} \lambda_{j\,1} ... \lambda_{j_k}$ are the Hessians of degree k of the eigenvalue vector $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n)$.

Definition. A twice smooth function $u \in C^2(D)$ is called $m-$convex in $D \subset \mathbb{R}^n$, $u \in m-cv(D)$, if its eigenvalue vectors $\lambda = \lambda(x) = (\lambda_1(x), \lambda_2(x), ..., \lambda_n(x))$ satisfy the conditions

$$m - cv \bigcap C^2(D) = \{H_k(u) = H_k(\lambda(x)) \geq 0, \; k = 1, ..., n - m + 1\}, \qquad (1)$$

at each point $o \in D$.

Note, that class $1 - cv$ coincides with the class of convex functions, and class $n - cv$ coincides with the class of subharmonic sh functions. Moreover, $1 - cv \subset\subset 2 - cv \subset ....$

The principle difficulties in the Theory of $m - cv$ functions are the introduction the class $m - cv \bigcap L^1_{loc}$, i.e. the definition of functions $m - cv(D)$ in the class of upper semi-continuous, locally integrable (or bounded) functions and the definition Hessians $H_k(u)$. So for $m = n$ the subharmonic functions in the class of upper semi-continuous, locally integrable functions $u(x) \in n - cv(D)$ is defined as a generalized function, where the Laplace operator $\Delta u = H_n(u)$ is a Borel measure.

2. To study $m - cv$ functions, we have established a suitable connection between the class of $m - cv$ functions and the well-known class of strongly $m-$ subharmonic (sh_m) functions. The class of sh_m functions has been well studied, the potential theory has been built in it and now it is a current direction in the theory of functions, has become the subject of research by many mathematicians (Z. Blocki [1], S. Dinev and S. Kolodziej [2], S. Li, H.Ch. Lu, A. Sadullaev, B. Abdullaev [3], etc.).

A twice smooth function $u(z) \in C^2(D), \quad D \subset \mathbb{C}^n$, is called strongly $m-$ subharmonic $u \in sh_m(D)$, if at each point of the domain D

$$\begin{gathered} sh_m(D) = \left\{ u \in C^2 : \ (dd^c u)^k \wedge \beta^{n-k} \geq 0, \quad k = 1, 2, ..., n-m+1 \right\} = \\ = \left\{ u \in C^2 : \ dd^c u \wedge \beta^{n-1} \geq 0, (dd^c u)^2 \wedge \beta^{n-2} \geq 0, ..., (dd^c u)^{n-m+1} \wedge \beta^{m-1} \geq 0 \right\}, \end{gathered} \tag{2}$$

where $\beta = dd^c \left\| z \right\|^2$ $-$is the standard volume form in $\mathbb{C}^n$.

Operators $(dd^c u)^k \wedge \beta^{n-k}$ are closely related to Hessians. For a doubly smooth function $u \in C^2(D)$, the second order differential $dd^c u = \frac{i}{2} \sum\limits_{j,k} \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} dz_j \wedge d\,\bar{z}_k$ (at a fixed point $o \in D$) is Hermitian quadratic form. After a suitable unitary coordinate transformation, it is reduced to diagonal form $dd^c u = \frac{i}{2} \left[\lambda_1 dz_1 \wedge d\,\bar{z}_1 + ... + \lambda_n dz_n \wedge d\,\bar{z}_n \right]$, where $\lambda_1 , ..., \lambda_n$ are the eigenvalues of the Hermitian matrix $(\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k})$, which are real: $\lambda = (\lambda_1 , ..., \lambda_n) \in \mathbb{R}^n$. Therefore, it is easy to see that

$$(dd^c u)^k \wedge \beta^{n-k} = k!(n-k)! H_k(u) \beta^n,$$

where $H_k(u) = \sum\limits_{1 \leq j_1 < ... < j_k \leq n} \lambda_{j_1} ... \lambda_{j_k}$ is the Hessian of degree k of the vector

$\lambda = \lambda(u) \in \mathbb{R}^n$.

Consequently, a doubly smooth function $u(z) \in C^2(D),\ \ D \subset \mathbb{C}^n$, is strongly $m-$subharmonic if at each point $o \in D$ we have

$$H_k(u) \geq 0,\ \ k = 1, 2, ..., n - m + 1. \tag{3}$$

This circumstance allows us to establish the following connection between $m - cv$ and sh_m. To do this, we embed $\mathbb{R}^n_x$ in $\mathbb{C}^n_z$, $\mathbb{R}^n_x \subset \mathbb{C}^n_z = \mathbb{R}^n_x + \mathbb{R}^n_y$ $(z = x + iy)$, as a real $n-$ dimensional subspace of the complex space $\mathbb{C}^n_z$.

Theorem 1. A twice smooth function $u(z) \in C^2(D),\ \ D \subset \mathbb{C}^n$, is called strongly subharmonic $u \in sh_m(D)$, if at each point of the domain D

$$\begin{gathered} sh_m(D) = \left\{u \in C^2 :\ (dd^c u)^k \wedge \beta^{n-k} \geq 0,\ \ k = 1, 2, ..., n - m + 1\right\} = \\ = \left\{u \in C^2 :\ dd^c u \wedge \beta^{n-1} \geq 0, (dd^c u)^2 \wedge \beta^{n-2} \geq 0, ..., (dd^c u)^{n-m+1} \wedge \beta^{m-1} \geq 0\right\}, \end{gathered} \tag{4}$$

where $\beta = dd^c \left\|z\right\|^2$ $-$is the standard volume form in $\mathbb{C}^n$.

Note that the concept of a strongly $m-$subharmonic function in the generalized sense is defined also in the class of bounded upper semi-continuous functions. In particular, for bounded functions $sh_m \bigcap L^{\infty}_{loc}$ operators $(dd^c u)^k \wedge \beta^{n-k} \geq 0,\ \ k = 1, 2, ..., n - m + 1$, are defined as a Borel measures.

Theorem 1 gives us the opportunity to define $m - cv$ functions in the class of L^1_{loc} functions and to define Hessians $H_k(u) \geq 0, k = 1, 2, ..., n - m + 1$ in the class of bounded $m - cv$ functions.

Theorem 2. *Let $u(x)$ be a bounded, $m-$ convex function in $D \subset \mathbb{R}^n_x$. Let us define Borel measures $\mu_k = (dd^c u)^k \wedge \beta^{n-k} \geq 0,\ k = 1, 2, ..., n - m + 1$ in $D \times \mathbb{R}^n_y \subset \mathbb{C}^n_z$. Then for any Borel sets $E_x \subset D, E_y = \mathbb{R}^n_y$ the measures $\frac{1}{mesE_y}\mu_k\left(E_x \times E_y\right)$ do not depend on the set $E_y \subset \mathbb{R}^n_y$, i.e. $\frac{1}{mesE_y}\mu_k\left(E_x \times E_y\right) = \nu_k\left(E_x\right)$.*

The measure ν_k is called as Hessians $H_k(u) = \nu_k, k = 1, 2, ..., n-m+1$. For a twice smooth function $u(z) \in m - cv(D) \bigcap C^2(D)$, the Hessians are ordinary functions; however, for a non-doubly smooth, but bounded semicontinuous function $u(z) \in m-cv(D) \bigcap L^{\infty}(D)$, the Hessians $H_k(u),\ k = 1, 2, ..., n-m+1$ are positive Borel measures.

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The direct scattering problem for general stationary systems on a semi-axis

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We consider the following system of first-order ordinary differential equations on the semi-axis $x \geq 0$:

$$-i\frac{dy_k(x)}{dx}+\sum_{j=1}^{n}c_{kj}(x)\,y_j(x)=\lambda\xi_k y_k(x)\,,\quad k=\overline{1,n} \tag{1}$$

where the coefficients $c_{kj}(x)\,,\ k,j=\overline{1,n}$ satisfy the conditions

$$|c_{kj}(x)\,|\le ce^{-\varepsilon x},\ c>0,\ \varepsilon>0 \tag{2}$$

$c_{jj}(x)=0,\ k,j=\overline{1,n},\quad \xi_1>\cdots>\xi_{n-3}>0>\xi_{n-2}>\cdots>\xi_n$.

We consider $(n-3)$ problems on the semi-exis. The k-th problem is the following:

$$y_j^k(x,\lambda)=A_je^{i\lambda\xi_jx}+o(1)\,,\ \ k=\overline{1,n-3},\ j=\overline{1,n-3},\ x\to+\infty \tag{3}$$

$$\begin{gathered}y_n^k(0,\lambda)=y_k^k(0,\lambda)\,,\\ y_{n-1}^k(0,\lambda)=\sum_{\substack{i=1\\ i\neq k}}^{n-3}y_i^k(0,\lambda)\\ y_{n-2}^k(0,\lambda)=\sum_{i=1}^{n-4}y_i^k(0,\lambda)\},\ k=\overline{1,n-3},\end{gathered} \tag{4}$$

here $n-3>3$, that is $n>6$.

The problem (1), (3), (4) is called the scattering problem on a semi-axis. İt is equivalent to the following system of integral equations:

$$\begin{gathered}y_p^k(x,\lambda)=A_pe^{i\lambda\xi_px}-i\sum_{j=1}^{n}\int_x^{+\infty}c_{pj}(t)\,y_j^k(t,\lambda)\,e^{i\lambda\xi_p(x-t)}dt,\ p=\overline{1,n-3}\\ y_p^k(x,\lambda)=B_p^ke^{i\lambda\xi_px}-i\sum_{j=1}^{n}\int_x^{+\infty}c_{pj}(t)\,y_j^k(t,\lambda)\,e^{i\lambda\xi_p(x-t)}dt,\ p=n-2,n-1,n.\end{gathered} \tag{5}$$

Where $B_p^k\ \left(p=n-2,n-1,n;\ k=\overline{1,n-3}\right)$ can be found by the boundary conditions (4).

The system (5) is the Volterra system. So for the real $\lambda\,(Im\lambda=0)$ from the conditions (2) it follows that this system has a unique bounded solution. That is the scattering problem for the system (1) has a unique bounded solution.

In the class of bounded functions for real λ from (2) it follows the correctness of equalities:

$$y_p^k(x,\lambda)=B_p^ke^{i\lambda\xi_px}+o(1)\,,\ x\to+\infty,\ p=\overline{n-2,n}\ k=\overline{1,n-3}. \tag{6}$$

From the asymptotics (6) based on the uniqueness of the solution of the scattering problem we can define rectangular matrices

$$S_k(\lambda):\begin{pmatrix} A_1\\ \vdots\\ A_{n-3}\end{pmatrix}\to\begin{pmatrix} B_{n-2}^k\\ B_{n-1}^k\\ B_n^k\end{pmatrix}\ ,k=\overline{1,n-3}$$

converting incident waves $\left\{A_1e^{i\lambda\xi_1x},\,A_2e^{i\lambda\xi_2x},\dots,A_{n-3}e^{i\lambda\xi_{n-3}x}\right\}^t$ to the scattering ones $\left\{B_{n-2}e^{i\lambda\xi_{n-2}x},\,B_{n-1}e^{i\lambda\xi_{n-1}x},\dots,B_ne^{i\lambda\xi_nx}\right\}^t$ (t here means transposition).

Matrices $S_k(\lambda)$ $\left(k=\overline{1,n-3}\right)$ have the form:

$$S_k(\lambda)=\begin{pmatrix} S_{11}^k(\lambda) & S_{12}^k(\lambda)\dots S_{1,n-3}^k(\lambda)\\ S_{21}^k(\lambda) & S_{22}^k(\lambda)\dots S_{2,n-3}^k(\lambda)\\ S_{31}^k(\lambda) & S_{32}^k(\lambda)\dots S_{3,n-3}^k(\lambda)\end{pmatrix}. \tag{7}$$

Matrix function $S(\lambda)=(S_1(\lambda),S_2(\lambda),\dots,S_{n-3}(\lambda))$ we will call the scattering problem for the system (1) on a semi-axis (that is the direct scattering problem).

The inverse scattering problem for the system (1) is to recover the coefficients of equations by given scattering matrix $S(\lambda)$.

Here assuming the absence of a singular spectrum (the problem's zeros are absent) the inverse problem comes to the scattering problem on the whole axis [1].

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On invariant deferred sequence spaces defined by modulus function

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Let σ be a one-to-one mapping from the set of natural numbers into itself. A continuous linear functional ϕ on l_∞ is said to be an invariant mean or a σ-mean if and only if

1) $\phi(x) \geq 0$ when the sequence $x = (x_k)$ has $x_n \geq 0$ for all n;
2) $\phi(e) = 1$ where $e = (1, 1, 1, \ldots)$ and
3) $\phi(x_{\sigma(n)}) = \phi(x)$ for all $x \in l_\infty$.

For certain class of mapping σ every invariant mean φ extends the limit functional on space c, in the sense that $\varphi(x) = \lim x$ for all $x \in c$.

Consequently, $c \subset V_\sigma$ where V_σ is the subset of all bounded sequences whose σ-means are equal.

In this paper, using non-negative summability matrix and invariant mean we present the notion of deferred $\sigma-$sequences spaces and further we have some inclusion relations among these sequence spaces.

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Compactness criterion of the weighted endomorphism of the ball-algebra $A(B^n)$

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In terms of both topological and algebraic properties of spaces with topological algebraic structure, most of the transformation processes that preserve their internal structure are in the form of composition operators $f \mapsto f \circ \varphi$. These operators correspond to their endomorphisms in the case of uniform algebras [3]. Therefore, the study of spectral properties of composition operators in uniform spaces and weighted (weighted type) endomorphisms in uniform algebras is significant. In this thesis, we investigate a necessary and sufficient condition for the compactness of a weighted endomorphism acting on a uniform algebra $A(B^n)$. Here $A(B^n)$ is a uniform algebra of functions that are analytic in the unit ball $B^n = \{z = (z_1, z_2, ..., z_n \in C^n|) \sum_{k=1}^{n} |z_k|^2 \ < 1\}$ and continuous on its closure. And B^n is the unit ball of C^n n-dimensional complex space. These endomorphisms are of the form $T : A(B^n) \longrightarrow A(B^n)$, $f \mapsto u \cdot (f \circ \varphi), \varphi : B^n \to B^n, u : B^n \to C$. Here, the map and the function u are analytic. Considering the work [1] and [2] for weighted composition operators and weighted endomorphisms, we get the following criterion for the compactness of the operator T.

Theorem 1. *A necessary and sufficient condition for the compactness of the endomorphism T generated by the function u and the map φ on $A(B^n)$ is either $\varphi = const$ or $\|\varphi(z)\| < 1$ for all $z \in S(u)$*

Here, S(u) is the support set of function u.

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Fourier multipliers on noncommutative Euclidean spaces

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In this work, we study Fourier multipliers on quantum Euclidean spaces and obtain results on their $L_p - L_q$ boundedness. On the way to get these results, we prove Paley, Hausdorff-Young, Hausdorff-Young-Paley, and Hardy-Littlewood inequalities on the quantum Euclidean space. As applications, we establish the $L_p - L_q$ estimate for the heat semigroup and Sobolev embedding theorem on quantum Euclidean spaces. We also obtain quantum analogues of logarithmic Sobolev and Nash type inequalities. This is a joint work with the Professor M. Ruzhansky and S. Shaimardan.

Isoabelian operators II: characterization

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We begin with various characterization of isoaberian operators in $B\,(X)$. They motivate famious indications of unitary operators in Hilbert space.

For the formulation of the theorems we need the following definitions reversibly operator $T \in B(X)$ is called isobelian if there is a semi-inner product $[.]$ compatable with the norm in X and such that for $x, y \in X : [Tx, y] = [x, T^{-1}y]$. Operator $T \in B(X)$ norm-unitary if T is invertible and $\|T\| = \|T^{-1}\| = 1$ unimodular if spectrum $\sigma(T)$ lies in unit circle $S^1 = \{z \in C : |z| = 1\}$; paranormal if $\|Tx\|^2 \leq \|T^2x\|$ for any $x \in S(X)$. The spectral radius $r(T)$ of operator $T \in B(X)$ is defined by equality $r(T) = \{|\lambda| : \lambda \in \sigma(T)\}$.

First part following theorem give reverse to result from [3, p. 511], showing that isoabelity of $T \in B(X)$ is independent of chose semi-inner product and dependence only from norm in X.

In second part of theorem statement of [4] in Hillert space onto any Banach space is transfered.

Third part of theorem sextension under consideration in [5] class of Banach spaces.

Theorem *Isoabelity of operator $T \in B(X)$ is equivalent any following conditions*
a) T is norm-unitary;
b) T is unimodular paranormal;
c) in smooth reflexive rotund X equality $T^{-1} = T^{+}$ is true where T^{+} is generalize adjoint for reversibly operator $T \in B(X)$.

Skheme of the proof. a) It using result of Koehler and Rosental $T \in B(X)$ in any Banach space X is an isometry if only if there exisist a semi-inner product determining the norm of X such that preserve , that is $[Tx, Ty] = [x, y]$ for all $x, y \in X$.

b) Unimodular paranormality of isoabelian operats it is evidently. Reverse statement it follows from equalities $r(T) = \|T\|$ and $r(T^{-1}) = \|T^{-1}\|$ by using article a) present theorem.

c) The proof of the statement is obtained with the help of Theorem [7, p. 210] about Riesz-Frechet generalized representation of linear functional in smooth reflexive rotund Banach space.

A note a few evident consequences of this theorem, which useful in further.

Corollary. *Isoabelian operators is form subgroup in group of reversible operators and isoabelity preserve under Banach conjugation of operators. Isoablian operators T is power-bounded, id est there exist a constant $M > 0$ such that $\|T^n\| \leq M$ for every positive integer n .*

As special case of theorem we have the validity of the following statement

Corollary. *a) If $T \in B(H)$ in Hilbert space H, then theorem a) be classical result: class of unistary operators is all automorphisms of space H [p.336] b) Unimodular paranormal operator $T \in B(X)$ is a unitary [4]*
c) In any smooth in uniformly rotund space X operator $T \in B(X)$ is generalize unitary if only if $T^{-1} = T^{+}$.

Proof. It is obvious uniformly rotund is rotund and in accordance with of Milman's-Pettis [2, p.510] uniformly rotund space is reflexive. As is know Day's exemple showing shows that class uniformly rotund spaces is strictly narrower than the class of reflexive rotund spaces.

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FUNCTION THEORY

An analogue of the Sadullaev-Dinew theorem for the class of m-convex functions.

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The report is devoted to $m-$convex functions, which are defined using operators in Hessians. Let $D \subset \mathbb{R}^n$ and $u(x) \in C^2(D)$. The matrix $\left(\frac{\partial^2 u}{\partial x_k \partial x_t}\right)$ is symmetric, $\frac{\partial^2 u}{\partial x_k \partial x_t} = \frac{\partial^2 u}{\partial x_t \partial x_k}$. Therefore, after a suitable orthonormal transformation, it is transformed into a diagonal form,

$$\left(\frac{\partial^2 u}{\partial x_k \partial x_t}\right) \to \begin{pmatrix} \lambda_1 & 0 & ... & 0 \\ 0 & \lambda_2 & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & \lambda_n \end{pmatrix},$$

where $\lambda_j = \lambda_j(x) \in \mathbb{R}$ are the eigenvalues of the matrix $\left(\frac{\partial^2 u}{\partial x_k \partial x_t}\right)$. Then

$$H^s(u) = H^s(\lambda) = \sum_{1 \le j_1 < ... < j_s \le n} \lambda_{j_1} ... \lambda_{j_s}$$

is called the Hessian of degree s of the eigenvalue vector $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n)$.

Definition 1. (see [10]) A twice smooth function $u \in C^2(D)$ is called m-convex in $D \subset \mathbb{R}^n$, $u \in m - cv(D)$, if its eigenvalue vector $\lambda = \lambda(x) = (\lambda_1(x), \lambda_2(x), ..., \lambda_n(x))$ satisfies the conditions

$$H^s(\lambda(x)) \geq 0, \ \forall x \in D, \ \ s = 1, ..., n - m + 1.$$

It is clear that $1 - cv(D) \subset m - cv(D) \subset n - cv(D) = sh(D)$. The theory of $m - cv$ functions is a new direction in the theory of real geometry. However, for $m = n$ the class $n - cv \cap C^2(D) = \{\lambda_1 + \lambda_2 + ... + \lambda_n \geq 0\}$ coincides with the class of subharmonic functions, and for $m = 1$ this class

$1-cv\cap C^2(D) = \{H^1(\lambda) \geq 0, ..., H^n(\lambda) \geq 0\} = \{\lambda_1 \geq 0, ..., \lambda_n \geq 0\}$ coincides with convex functions in $\mathbb{R}^n$. The class of convex functions has been well studied by A. Aleksandrov, I. Bakelman, A. Pogorelov, A. Artikbaev in works [1]-[4]. When $m > 1$ the class of functions has been studied in a series of works by N.Ivochkina, N. Trudinger, X.Wang and others (see [9]-[14]). We study $m-cv(D)$ functions based on new approaches. Namely, through their connection with the well-known sh_m functions: Let us embed $\mathbb{R}^n_x$ into $\mathbb{C}^n_z$, $\mathbb{R}^n_x \subset \mathbb{C}^n_z = \mathbb{R}^n_x + i\mathbb{R}^n_y\,(z = x + iy)$, as a real $n-$dimensional subspace of the complex space $\mathbb{C}^n_z$.

Theorem 1 (see [8]). *A twice smooth function $u(x) \in C^2(D)$, $D \subset \mathbb{R}^n_x$, is $m-cv$ in D if and only if a function $u^c(z) = u^c(x+iy) = u(x)$, that does not depend on variables $y \in \mathbb{R}^n_y$, is sh_m in the domain $D \times \mathbb{R}^n_y$.*

To construct the Potential theory in the class of $m-sh$ functions, B. Abdullaev introduced the class

$$(A)sh_m(D) = \left\{ u \in C^2(D),\ dd^c u \wedge \left(dd^c|z|^2\right)^{m-1} \geq 0,\ (dd^c u)^{n-m+1} \wedge \left(dd^c|z|^2\right)^{m-1} \geq 0 \right\}.$$

In the work [7], A. Sadullaev proved that for $m = 2$ this class coincides with the class $sh_m(D)$. He also posed the problem of whether these two classes coincide at all? In the work [6], S.Dinew showed that for $n \leq 7$ these classes actually coincide, but for $n = 11$, $m = 3$ these classes are different. In the work [5], the authors somewhat improved Dinew's result, proving that for $n = 10$, $m = 3$ Sadullaev's problem does not apply, but for $n = 9$, $n = 8$ and $m = 3$ this problem is true. Using Theorem 1, we will prove an analogue of these results for the classes $m-cv$ and $(A)m-cv(D)$ functions: A twice smooth function $u \in C^2(D)$ is called $(A)m-cv$ in $D \subset \mathbb{R}^n$, if its eigenvalue vector $\lambda = \lambda(x) = (\lambda_1(x), \lambda_2(x), ..., \lambda_n(x))$ satisfies the conditions $\lambda_{j_1}(x) + \lambda_{j_2}(x) + ... + \lambda_{j_m}(x) \geq 0$, $1 \leq j_1 < ... < j_m \leq n$, $H^{n-m+1}(\lambda(x)) \geq 0$, $\forall x \in D$.

Theorem 2. *The classes $m-cv$ and $(A)m-cv$ coincide when $m = 2$. For $n \leq 7$ they are the same for all $2 \leq m \leq n$. When $n = 10$, $m = 3$ they do not coincide. However, for $n = 9$, $n = 8$ and $m = 3$ they are the same.*

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On the Markoff inequality for algebraic polynomials in a weighted Lebesgue space in regions of complex plane

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Let $G \subset \mathbb{C}$ be a bounded Jordan region with boundary $L := \partial G$. Let $\wp_n$ denotes the class of all algebraic polynomials $P_n(z)$ of degree at most $n \in \mathbb{N}$; $h(z)$ be a generalized Jacobi weight function.

Let $0 < p \leq \infty$. For the arbitrary Jordan region G with rectifiable boundary L, we introduce:

$$\begin{aligned} \|P_n\|_p \ &: = \|P_n\|_{\mathcal{L}_p(h,L)} := \left(\int\limits_L h(z) \left|P_n(z)\right|^p |dz| \right)^{1/p} < \infty, \ 0 < p < \infty, \\ \|P_n\|_\infty \ &: = \|P_n\|_{\mathcal{L}_\infty(1,L)}, \ p = \infty. \end{aligned}$$

In this work, we study the following type estimates for the regions having cusps

$$\|P_n^{(m)}\|_\infty \leq \lambda_n \|P_n\|_p, \tag{15}$$

where $\lambda_n := \lambda_n(L, h, m, p) > 0$, $\lambda_n \to \infty$, $n \to \infty$, is a constant, depending on the geometrical properties of the curve L and the weight function h.

Inequalities of type (15) for some $m \geq 0$, $h(z)$, L and $0 < p \leq \infty$, have been studied by many mathematicians since the beginning of the 20th century (Bernstein SN., Szegö G & Zygmund A.). Over the past few years, such inequalities for various spaces have been studied by, for example, Nikol'skii SM., Milovanovic GV & Mitrinovic DS. & Rassias ThM., Dzjadyk VK, Andrashko, Mamedkhanov DI., Pritsker I., Andrievskii VV., Ditzian Z & Tikhonov S., Ditzian Z & Prymak A., Nevai P &TotikV., Abdullayev FG and etc (extensive reference can be found in [1], [2] and the references cited therein).

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Absolute convergence of the multiple series of Fourier-Haar coefficients

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The problems related to the behavior of a single series of Fourier-Haar are well studied [8]. Namely, P. Ulianov [14] and B. Golubov [7] received the results related to the problems of absolute convergence of the series of Fourier-Haar coefficients. Some generalizations of these results related were received by Z. Chanturia [3], T. Akhobadze [1], U. Goginava [6] and by the author [2]. In the term of modulus of smoothness the problem of absolute convergence of the series of Fourier-Haar coefficients was studied by V. Krotov [11]. Multidimensional analogies corresponding to the results of V. Krotov were formulated in the works of V. Tsagareishvili [13] and G. Tabatadze [12].

In 1881 C. Jordan [10] introduced a class of functions of bounded variation and, applying it to the theory of Fourier series proved that if a continuous function has bounded variation, then its Fourier series converges uniformly. In 1906 G. Hardy [9] generalized the Jordan criterion to double Fourier series and introduced the notion of bounded variation (HBV) for a function of two variables. He proved that if a continuous function of two variables has bounded variation (in the sense of Hardy), then its Fourier series converges uniformly in the sense of Pringsheim. In 1999 U. Goginava [4] introduced the notion of bounded partial p-variation (PBV_p) and proved that the class $PBV_p(p \geq 1)$ guarantees the uniform convergence of N-dimensional trigonometric Fourier series of the continuous function.

Let us denote Fourier-Haar multiple coefficients of the function $f \in L([0,1])^N$ by $C_{\overrightarrow{m}}(f)$, i.e

$$C_{\overrightarrow{m}}(f) = \int\limits_{[0,1]^N} f \cdot \chi_{\overrightarrow{m}} d\overrightarrow{x}.$$

We study the problem of absolute convergence of the series of Fourier-Haar coefficients for the classes of functions with bounded partial p-variations, which were first considered by U. Goginava(see [4] for $p = 1$ and [5] for $p > 1$).

Definition: Let f be a function defined on $[0,1]^N$ and 1$-$periodic with respect to each variable. f is said to be a function of bounded partial p-variation $\left(f \in PBV_p(I^N)\right)$, if for any $i = 1, 2, ..., N$ and $n = 1, 2, ...$

$$V_i(f) = \sup_{x_j, j \in \{1,...,N\} \setminus \{i\}} \sup_{\Pi} \sum_{k=0}^{n-1} \left| f\left(x_1, ..., x_{i-1}, x_i^{(2k)}, x_{i+1}, ..., x_N\right) - \right.$$

$$\left. - f\left(x_1, ..., x_{i-1}, x_i^{(2k+1)}, x_{i+1}, ..., x_N\right)\right|^p < \infty,$$

where Π is an arbitrary system of disjoint intervals $\left(x_i^{(2k)}, x_i^{(2k+1)}\right)$ $(k = 0, 1, ..., n-1)$ on $[0,1]$, i.e.

$$0 \leq x_i^{(0)} < x_i^{(1)} < x_i^{(2)} < ... < x_i^{(2n-2)} < x_i^{(2n-1)} \leq 1.$$

The following theorems are true:

Theorem 1. *Let $f \in PBV_p(I^N), p \geq 1$ and $\beta > \frac{2pN}{2+pN}$. Then*

$$\sum_{n_1=0}^{\infty} \cdots \sum_{n_N=0}^{\infty} |C_{n_1,...,n_N}(f)|^\beta < \infty.$$

Theorem 2. *Let $f \in PBV_p(I^N), p \geq 1$ and $\alpha < \frac{1}{pN} - \frac{1}{2}$. Then*

$$\sum_{n_1=0}^{\infty} \cdots \sum_{n_N=0}^{\infty} \prod_{i=1}^{N} (n_i+1)^\alpha |C_{n_1,...,n_N}(f)| < \infty.$$

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On properties of the Riesz transform of Lebesgue integrable functions

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The j-th Riesz transform of the function $f \in L_p(\mathbb{R}^d)$, $1 \le p < +\infty$ is defined as the following singular integral:

$$R_j(f)(x) = \gamma_{(d)} \cdot \lim_{\varepsilon\to 0+} \int_{\{y\in\mathbb{R}^d:\ |x-y|>\varepsilon\}} \frac{x_j - y_j}{|x-y|^{d+1}} f(y)dy,$$

where $\gamma_{(d)} = \frac{\Gamma((d+1)/2)}{\pi^{(d+1)/2}}$, $\Gamma(z) = \int_0^{+\infty} t^{z-1}\mathrm{e}^{-t}dt$ is Euler's Gamma function.

It is well known (see [1]) that the Riesz transform plays an important role in the theory of harmonic functions. The boundary values of harmonically conjugate in the upper half space functions are interconnected by the Riesz transform.

From the theory of singular integrals (see [1]) it is known that the Riesz transform is a bounded operator in the space $L_p(\mathbb{R}^d)$, $1 < p < \infty$, that is, if $f \in L_p(\mathbb{R}^d)$, then $R_j(f) \in L_p(\mathbb{R}^d)$ and the inequality

$$\|R_j f\|_{L_p} \le C_p \|f\|_{L_p} \tag{16}$$

holds; in the case $f \in L_1(\mathbb{R}^d)$ only the weak inequality holds:

$$m\{x \in R^d :\ |(R_j f)(x)| > \lambda\} \le \frac{C_1}{\lambda} \|f\|_{L_1}, \tag{17}$$

where m stands for the Lebesgue measure, C_p, C_1 are constants independent of f. From the inequalities (16), (17) it follows that the Riesz transform of the function $f \in L_p(\mathbb{R}^d)$ satisfies the condition

$$m\{x \in \mathbb{R}^d : \ |(R_j f)(x)| > \lambda\} = o(1/\lambda), \quad \lambda \to +\infty.$$

Note that the Riesz transform of a function $f \in L_1(\mathbb{R}^d)$ is generally not Lebesgue integrable. In this paper using the notions of A-integral and Q-integral, introduced by E. Titchmarsh, we prove the analogue of Riesz's equation and Titchmarsh's equation for the Riesz transform of a Lebesgue integrable function.

For a measurable complex-valued function f on $\mathbb{R}^d$ we set

$$[f(x)]_{\delta,\lambda} = [f(x)]^{\delta,\lambda} = f(x) \ \text{ for } \ \delta \le |f(x)| \le \lambda,$$
$$[f(x)]_{\delta,\lambda} = [f(x)]^{\delta,\lambda} = 0 \ \text{ for } \ |f(x)| < \delta,$$
$$[f(x)]_{\delta,\lambda} = \lambda \cdot \operatorname{sgn} f(x), \quad [f(x)]^{\delta,\lambda} = 0 \ \text{ for } \ |f(x)| > \lambda,$$

where $\operatorname{sgn} w = \frac{w}{|w|}$ for $w \neq 0$ and $\operatorname{sgn} 0 = 0$. In 1929, E.Titchmarsh [2] introduced the notions of Q- and Q'-integrals.

Definition 1. If the finite limit

$$\lim_{\substack{\delta \to 0+ \\ \lambda \to +\infty}} \int_{\mathbb{R}^d} [f(x)]_{\delta,\lambda}\, dx \quad \left(\lim_{\substack{\delta \to 0+ \\ \lambda \to +\infty}} \int_{\mathbb{R}^d} [f(x)]^{\delta,\lambda}\, dx \right)$$

exists, then f is said to be *Q-integrable* (*Q'-integrable*, respectively) on $\mathbb{R}^d$, and this fact is written as $f \in Q(\mathbb{R}^d)$ ($f \in Q'(\mathbb{R}^d)$, respectively). The value of this limit is referred to as *Q-integral* (*Q'-integral*, respectively) of the function f and is denoted by

$$(Q) \int_{\mathbb{R}^d} f(x) dx \left((Q') \int_{\mathbb{R}^d} f(x) dx \right).$$

A very uncomfortable fact impeding application of Q and Q'-integrals, when studying diverse problems of function theory, is the absence of the additivity property, that is, the Q-integrability (Q'-integrability) of two functions does not imply the Q-integrability (Q'-integrability) of their sum. If one adds the conditions

$$m\{x \in \mathbb{R}^d : \ |f(x)| > \delta\} = o(1/\delta), \quad \delta \to 0+, \tag{18}$$

$$m\{x \in \mathbb{R}^d : \ \ |f(x)| > \lambda\} = o(1/\lambda), \quad \lambda \to +\infty, \tag{19}$$

to the definition of Q-integrability (Q'-integrability), then Q-integral and Q'-integral coincide ($Q(\mathbb{R}^d) = Q'(\mathbb{R}^d)$) and these integrals become additive.

Definition 2. If $f \in Q'(\mathbb{R}^d)$ (or $f \in Q(\mathbb{R}^d)$) and the conditions (18) and (19) hold, then f is said to be A-*integrable* on $\mathbb{R}^d$, and this fact is written as $f \in A(\mathbb{R}^d)$. The limit $\lim\limits_{\substack{\delta\to 0+\\ \lambda\to+\infty}} \int_{\mathbb{R}^d} [f(x)]^{\delta,\lambda} dx$ (or the limit $\lim\limits_{\substack{\delta\to 0+\\ \lambda\to+\infty}} \int_{\mathbb{R}^d} [f(x)]_{\delta,\lambda} dx$) is denoted in this case by

$$(A)\int_{\mathbb{R}^d} f(x)\, dx.$$

Theorem 1. *If $f \in L_1(\mathbb{R}^d)$ and the bounded function $g \in L_p(\mathbb{R}^d)$, $1 < p < \infty$, is such that its Riesz transforms $R_j(g)$, $j = 1, ..., d$, are also bounded in $\mathbb{R}^d$, then for any $j = 1, ..., d$ the function $g \cdot R_j(f)$ is A-integrable on $\mathbb{R}^d$ and*

$$(A)\int_{\mathbb{R}^d} g(x) \cdot (R_j f)(x)dx = -\int_{\mathbb{R}^d} f(x) \cdot (R_j g)(x)dx.$$

Theorem 2. *Let $f \in L_1(\mathbb{R}^d)$. Then for any $j = 1, ..., d$ the function $R_j(f)$ is Q'-integrable on $\mathbb{R}^d$ and*

$$(Q')\int_{\mathbb{R}^d} (R_j f)(x)dx = 0.$$

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On the equioscillation theorem for approximation by sums of two algebras

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The classical Chebyshev equioscillation or alternation theorem gives a criterion for a polynomial P of degree not greater than n to be the best uniform approximation to a continuous real valued function f, using the oscillating nature of the difference $f - P$. More precisely, the theorem asserts that P is the best uniform approximation to f on $[0, 1]$ if and only if there exist $n + 2$ points t_i in $[0, 1]$ such that

$$f(t_k) - P(t_k) = (-1)^k \max_{t\in[0,1]} |f(t) - P(t)|\,,\ k = 1, ..., n + 2.$$

We obtain a Chebyshev type equioscillation theorem for approximation of a continuous function, defined on a compact metric space, by a sum of two algebras. To make the problem more precise, assume X is a compact metric space, $C(X)$ is the space of real-valued continuous functions on X, A_1 and A_2 are closed subalgebras of $C(X)$ containing constants. For a given function $f \in C(X)$ consider the approximation of f by elements of $A_1 + A_2$. We ask and answer the following question: which conditions imposed on $u_0 \in A_1 + A_2$ are necessary and sufficient for the equality

$$\|f - u_0\| = \inf_{u\in A_1+A_2} \|f - u\|? \tag{1}$$

Here $\|\cdot\|$ denotes the standard uniform norm in $C(X)$. Recall that a function u_0 satisfying (1) is called a best approximation to f.

It should be remarked that approximation problems concerning sums of algebras were studied in many papers (see Khavinson's monograph [5] for an extensive discussion).

Note that the algebras A_i, in particular cases, turn into algebras of univariate functions, ridge functions and radial functions. The literature abounds with the use of ridge functions and radial functions. Ridge functions and radial functions are defined as multivariate functions of the form $g(\mathbf{a}\cdot\mathbf{x})$ and $g(|\mathbf{x}-\mathbf{a}|_e)$ respectively, where $\mathbf{a}\in\mathbb{R}^d$ is a fixed vector, $\mathbf{x}\in\mathbb{R}^d$ is the variable, $\mathbf{a}\cdot\mathbf{x}$ is the usual inner product, $|\mathbf{x}-\mathbf{a}|_e$ is the Euclidean distance between $\mathbf{x}$ and $\mathbf{a}$, and g is a univariate function.

Define the equivalence relation R_i, $i=1,2$, for elements in X by setting

$$a \overset{R_i}{\sim} b \text{ if } f(a)=f(b) \text{ for all } f\in A_i.$$

Then, for each $i=1,2$, the quotient space $X_i=X/R_i$ with respect to the relation R_i, equipped with the quotient space topology, is compact and the natural projections $s: X\to X_1$ and $p: X\to X_2$ are continuous. In view of the Stone-Weierstrass theorem, the algebras A_1 and A_2 have the following set representations

$$\begin{aligned} A_1 &= \{g(s(x)):\ g\in C(X_1)\}, \\ A_2 &= \{h(p(x)):\ h\in C(X_2)\}. \end{aligned}$$

We proceed with the definitions of *bolts* and *extremal bolts*. These objects are essential for our analysis.

Definition 1. (see [6]) *A finite or infinite ordered set $l=\{x_1,x_2,...\}\subset X$, where $x_i\neq x_{i+1}$, with either $s(x_1)=s(x_2), p(x_2)=p(x_3), s(x_3)=s(x_4),...$ or $p(x_1)=p(x_2), s(x_2)=s(x_3), p(x_3)=p(x_4),...$ is called a lightning bolt (or, simply, a bolt) with respect to the algebras A_1 and A_2. If in a finite bolt $\{x_1,...,x_n,x_{n+1}\}$, $x_{n+1}=x_1$ and n is an even number, then the bolt $\{x_1,...,x_n\}$ is said to be closed.*

Definition 2. *A finite or infinite bolt $\{x_1,x_2,...\}$ is said to be extremal for a function $f\in C(X)$ if $f(x_i)=(-1)^i\left\|f\right\|, i=1,2,...$ or $f(x_i)=(-1)^{i+1}\left\|f\right\|$, $i=1,2,...$*

Our main result is the following theorem.

Theorem 1. *Assume X is a compact metric space. A function $u_0\in A_1+A_2$ is a best approximation to a function $f\in C(X)$ if and only if there exists a closed or infinite bolt extremal for the function $f-u_0$.*

The proof of Theorem 1 was given in [2]. In [1], Theorem 1 was proved under additional assumption that the algebras have the C-property, that is, for any $w\in C(X)$, the functions

$$\begin{aligned} g_1(a) &= \max_{\substack{x\in X\\ s(x)=a}} w(x),\ g_2(a) = \min_{\substack{x\in X\\ s(x)=a}} w(x),\ a\in X_1,\\ h_1(b) &= \max_{\substack{x\in X\\ p(x)=b}} w(x),\ h_2(b) = \min_{\substack{x\in X\\ p(x)=b}} w(x),\ b\in X_2 \end{aligned}$$

are continuous.

Note that in the special case when $X\subset\mathbb{R}^2$ and s,p are the coordinate functions, a Chebyshev type alternation theorem was first obtained by Khavinson [4]. In [3], similar alternation theorems were proved for ridge functions and certain function compositions.

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On complete elliptic integrals and their applications in certain problems of structural mechanics

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Motivated by the work of [3, 4, 5], several authors studied generalized trigonometric functions and established the inequalities and identities involving these functions. Making a contribution to the topic, we study generalized complete elliptic integrals of the first and second kind, defined by

$$\mathcal{K}_p(r) = \int_0^{\pi_p/2} \frac{dt}{(1 - r^p \sin_p^p t)^{1-1/p}} = \int_0^1 \frac{dt}{(1-t^p)^{1/p}(1-(rt)^p)^{1-1/p}}$$

$$\mathcal{E}_p(r) = \int_0^{\pi_p/2} (1 - r^p \sin_p^p t)^{1/p}\, dt = \int_0^1 \left(\frac{1-(rt)^p}{1-t^p}\right)^{1/p}, \; r \in (0,\, 1),\; p > 1,$$

respectively, As well, we establish two-sided sharp inequities involving theses integrals and their applications in some certain problems of mechanics. Moreover, we obtain some sharp functional inequalities for the generalized Grötzsch ring function.

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On Riesz type factorization for noncommutative Hardy spaces

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Let $0 < p \leq \infty$. L^p denotes the L^p-space relative to the circle group $\mathbb{T}$, which is equipped with its normalized Haar measure denoted as m. Furthermore, H^p represents the classical Hardy space consisting of analytic functions on the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ of the complex plane. It is a well-established fact that this space can be understood as a closed subspace of L^p, namely the closure in L^p (for $p = \infty$ we consider the weak*-closure) of the linear span of the functions $\{e^{it}|n \geq 0\}$. It is known as the Riesz factorization theorem: *Let $0 < p, q, r \leq \infty$ such that $1/p = 1/q + 1/r$. A function f is in H^p if and only if there exist $g \in H^q$ and $h \in H^r$ with $f = g.h$ and* $\|f\|_{H^p} = \|g\|_{H^q}\|h\|_{H^r}$. If we define

$$H^q \odot H^r = \{f : \ f = gh, \ g \in H^q, \ h \in H^r\}$$

and

$$\|f\|_{H^q \odot H^r} = \inf\{\|g\|_{H^q}\|h\|_{H^r} : \ f = gh, \ g \in H^q, \ h \in H^r\},$$

then the Riesz factorization theorem implies that

$$H^p = H^q \odot H^r.$$

In the finite case, the Riesz factorization theorem for noncommutative Hardy spaces has been systematically developed. Let $\mathcal{A}$ be a finite subdiagonal algebra in Arveson's sense. Let $H^p(\mathcal{A})$ be the associated noncommutative Hardy spaces, $0 < p \leq \infty$. If $0 < p, q, r \leq \infty$ such that $1/p = 1/q + 1/r$, then

$$H^p(\mathcal{A}) = H^q(\mathcal{A}) \odot H^r(\mathcal{A}). \tag{1}$$

Power in [2] obtained the Riesz weak factorization in noncommutative H^1 spaces associated with general nest subalgebras of semifinite factors. Our main purpose is to extend the Riesz-type weak factorization to symmetric quasi-Hardy spaces associated with semifinite subdiagonal algebras. The author and

Ospanov [1] proved that if $\mathcal{A}$ is a semifinite subdiagonal algebra in Arveson's sense and $H^p(\mathcal{A})$ is the associated noncommutative Hardy spaces $(0 < p \leq \infty)$, then for $0 < p, q, r \leq \infty$ such that $1/p = 1/q + 1/r$, the Riesz type weak factorization also holds in this case.

$E_1(\mathcal{A}) \boxdot E_2(\mathcal{A})$ is defined as the space of all operators x in $E_1 \odot E_2(\mathcal{A})$ for which there exist a sequence in $(a_n)_{n\geq 1}$ in $E_1(\mathcal{A})$ and a sequence $(y_n)_{n\geq 1}$ in $E_2(\mathcal{A})$ such that $\sum_{n\geq 1} \|a_n\|_{E_1}^r \|b_n\|_{E_2}^r < \infty$ and $x = \sum_{n\geq 1} a_n b_n$. For each $x \in E_1(\mathcal{A}) \boxdot E_2(\mathcal{A})$, we define

$$\|x\|^r_{E_1(\mathcal{A})\boxdot E_2(\mathcal{A})} = \inf \left\{ \sum_{n\geq 1} \|a_n\|^r_{E_1} \|b_n\|^r_{E_2} \right\}.$$

where the infimum runs over all possible factorizations of x as above. Then $\|\cdot\|_{E_1(\mathcal{A})\boxdot E_2(\mathcal{A})}$ is a quasi norm and $E_1(\mathcal{A}) \boxdot E_2(\mathcal{A})$ is a quasi Banach space.

It is clear that $E_1(\mathcal{A}) \boxdot E_2(\mathcal{A}) \subset E_1 \odot E_2(\mathcal{A})$ and

$$\|x\|_{E_1 \odot E_2} \leq C \|x\|_{E_1(\mathcal{A})\boxdot E_2(\mathcal{A})}, \quad \forall x \in E_1(\mathcal{A}) \boxdot E_2(\mathcal{A}). \tag{2}$$

Theorem 1. Let E_j be a symmetric quasi Banach function space on $(0, \alpha)$ with order continuous norm which is s_j-convex for some $0 < s_j < \infty$, $j = 1, 2$ and $0 < \theta < 1$. If $q_{E_1}, q_{E_2} < \infty$, then

$$E_1(\mathcal{A}) \boxdot E_2(\mathcal{A}) = E_1 \odot E_2(\mathcal{A}).$$

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About embedding of Nikol'skii-Besov spaces with a dominant mixed derivative and a mixed metric

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Let $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{R}^n_+$, we define the following set

$$\Omega_{\mathbf{a}} = \{\mathbf{x} = (x_1, \ldots, x_n) : a_j < x_j < 2a_j, j = 1, \ldots, n\}.$$

Let us consider an analogue of the Vallee-Poussin kernel for the parameter $\mathbf{M} = (M_1, \ldots, M_n) \in \mathbb{Z}^n$, which is defined by the following way

$$V_{\mathbf{M}}(\mathbf{t}) = \frac{1}{\prod_{j=1}^n M_j} \int_{\Omega_{\mathbf{M}}} \prod_{j=1}^n \frac{\sin \lambda_j t_j}{t_j} d\lambda.$$

Let $\mathbf{1} \le \mathbf{p} = (p_1, \ldots, p_n) \le \infty$ be multi-index. A Lebesgue space with a mixed metric $L_{\mathbf{p}}(\mathbb{R}^n)$ is a set of measurable functions for which the following norm is finite

$$\|f\|_{L_{\mathbf{p}}(\mathbb{R}^n)} = \left(\int_{-\infty}^{\infty} \left(\ldots \left(\int_{-\infty}^{\infty} |f(x_1, \ldots, x_n)|^{p_1} dx_1 \right)^{\frac{p_2}{p_1}} \ldots \right)^{\frac{p_n}{p_{n-1}}} dx_n \right)^{\frac{1}{p_n}}.$$

A generalized function f is called regular in the sense of $L_{\mathbf{p}}(\mathbb{R}^n)$ if for some $\rho_0 > 0$

$$I_{\rho_0} f = F \in L_{\mathbf{p}}(\mathbb{R}^n),$$

here

$$I_{\rho_0} f = \mathfrak{F}^{-1} \left(\left(1 + |\xi|^2\right)^{-\rho_0/2} \mathfrak{F}(f) \right),$$

and $\mathfrak{F}$ and $\mathfrak{F}^{-1}$ are the direct and inverse Fourier transforms, respectively.

Let further f be a regular function in the sense of $L_{\mathbf{p}}(\mathbb{R}^n)$, we assume

$$\sigma_{\mathbf{M}}(f) = \left(\frac{2}{\pi}\right)^{n/2} I_{-\rho}(V_{\mathbf{M}} * I_\rho f),$$

where $\rho > 0$ is large enough so that $I_\rho f \in L_{\mathbf{p}}(\mathbb{R}^n)$.

A regular expansion of a function f in the sense of $L_{\mathbf{p}}(\mathbb{R}^n)$ over Vallee-Poussin sums is the following representation $f = \sum_{\mathbf{s}\in\mathbb{Z}_+^n}{}' Q_{\mathbf{s}}(f)$, where $Q_{\mathbf{s}}(f) = \sigma_{2^{\mathbf{s}}}(f) - \sigma_{2^{\mathbf{s}-\mathbf{1}}}(f)$, and the sign $'$ in the summation operation means that if some index $s_j = 0$, then subtraction at the corresponding index in the expression for $Q_{\mathbf{s}}(f)$ is not performed.

Let further $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n$ and $\mathbf{1} \le \mathbf{q} = (q_1, \dots, q_n) \le \infty$. The Nikol'skii-Besov space $S_{\mathbf{p}}^{\alpha\mathbf{q}}B(\mathbb{R}^n)$ with a dominant mixed derivative and with a mixed metric is the set of regular in the sense of $L_{\mathbf{p}}(\mathbb{R}^n)$ functions f for which the following norm is finite

$$\|f\|_{S_{\mathbf{p}}^{\alpha\mathbf{q}}B(\mathbb{R}^n)} = \left\| \left\{ 2^{(\alpha,\mathbf{s})} \|Q_{\mathbf{s}}(f)\|_{L_{\mathbf{p}}(\mathbb{R}^n)} \right\}_{\mathbf{s}\in\mathbb{Z}_+^{\mathbf{n}}} \right\|_{l_{\mathbf{q}}},$$

here $(\alpha, \mathbf{s}) = \sum_{j=1}^{n} \alpha_j s_j$ is the inner product and $\|\cdot\|_{l_{\mathbf{q}}}$ is the norm of the discrete Lebesgue space $l_{\mathbf{q}}$ with a mixed metric.

Remark 1. *In the case when* $\alpha = (\alpha_1, \dots, \alpha_n) > \mathbf{0}$*, for the parameter* $\mathbf{q} = (\infty, \dots, \infty)$ *these spaces were introduced and studied in the works of [1, 2], and the case of* $\mathbf{p} = (p, \dots, p)$ *and* $\mathbf{q} = (q, \dots, q)$ *considered in the works of [3, 4, 5].*

The anisotropic Lorentz space $L_{\mathbf{pr}}(\mathbb{R}^n)$ is the set of functions such that the following quasi-norm is finite

$$\|f\|_{L_{\mathbf{pr}}(\mathbb{R}^n)} =$$

$$= \left(\int_0^\infty \left(\dots \left(\int_0^\infty \left(t_1^{\frac{1}{p_1}} \dots t_n^{\frac{1}{p_n}} f^{*_1,\dots,*_n}(t_1, \dots, t_n) \right)^{r_1} \frac{dt_1}{t_1} \right)^{\frac{r_2}{r_1}} \dots \right)^{\frac{r_n}{r_{n-1}}} \frac{dt_n}{t_n} \right)^{\frac{1}{r_n}},$$

where $f^*(\mathbf{t}) = f^{*_1,\dots,*_n}(t_1, \dots, t_n)$ is repeated non-increasing rearrangement of a function $f(\mathbf{x}) = f(x_1, \dots, x_n)$.

In this work, the embedding theorems for Nikol'skii-Besov spaces with a mixed derivative and mixed metric and anisotropic Lorentz spaces are proved.

Theorem 1. *Let* $-\infty < \alpha_0 = (\alpha_1^0, \dots, \alpha_n^0) < \alpha_1 = (\alpha_1^1, \dots, \alpha_n^1) < \infty$, $\mathbf{1} \le \tau = (\tau_1, \dots, \tau_n) \le \infty$ *and* $\mathbf{1} \le \mathbf{p}_0 = (p_1^0, \dots, p_n^0), \mathbf{p}_1 = (p_1^1, \dots, p_n^1) < \infty$.

Then the embedding

$$S_{\mathbf{p}_1}^{\alpha_1\tau}B(\mathbb{R}^n) \hookrightarrow S_{\mathbf{p}_0}^{\alpha_0\tau}B(\mathbb{R}^n)$$

holds for $\alpha_0 - 1/\mathbf{p}_0 = \alpha_1 - 1/\mathbf{p}_1$.

Theorem 2. *Let* $\mathbf{1} \leq \mathbf{p} = (p_1, \ldots, p_n) < \mathbf{q} = (q_1, \ldots, q_n) < \infty$ *and* $\mathbf{1} \leq \tau = (\tau_1, \ldots, \tau_n) \leq \infty$. *Then the embedding*

$$S_{\mathbf{p}}^{\alpha\tau}B(\mathbb{R}^n) \hookrightarrow L_{\mathbf{q}\tau}(\mathbb{R}^n)$$

holds for $\alpha = 1/\mathbf{p} - 1/\mathbf{q}$.

Theorem 3. *Let* $\mathbf{1} < \mathbf{q} = (q_1, \ldots, q_n) < \mathbf{p} = (p_1, \ldots, p_n) < \infty$ *and* $\mathbf{1} \leq \tau = (\tau_1, \ldots, \tau_n) \leq \infty$. *Then the embedding*

$$L_{\mathbf{q}\tau}(\mathbb{R}^n) \hookrightarrow S_{\mathbf{p}}^{\alpha\tau}B(\mathbb{R}^n),$$

holds for $\alpha = 1/\mathbf{p} - 1/\mathbf{q}$.

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On the analytic continuation of the Lauricella–Saran's hypergeometric function F_K

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Lauricella–Saran's hypergeometric function F_K is defined by triple power series (see, [1])

$$F_K(a_1, a_2, b_1, b_2; c_1, c_2, c_3; \mathbf{z}) = \sum_{p,q,r=0}^{+\infty} \frac{(a_1)_p (a_2)_{q+r} (b_1)_{p+r} (b_2)_q}{(c_1)_p (c_2)_q (c_3)_r} \frac{z_1^p z_2^q z_3^r}{p!q!r!},$$

where $a_1, a_2, b_1, b_2, c_1, c_2, c_3 \in \mathbb{C}$, $c_1, c_2, c_3 \notin \{0, -1, -2, \ldots\}$, $(\cdot)_k$ is the Pochhammer symbol defined for any complex number α and non-negative integer k by $(\alpha)_0 = 1$ and $(\alpha)_k = \alpha(\alpha+1)_{k-1}$, $\mathbf{z} = (z_1, z_2, z_3) \in \Omega_{F_K}$,

$$\Omega_{F_K} = \{\mathbf{z} \in \mathbb{C}^3 : \ |z_k| < 1, \ k = 1, 2, \ |z_3| < (1 - |z_1|)(1 - |z_2|)\}.$$

Theorem 1. *A function*

$$\frac{F_K(a_1, a_2, b_1, b_2; a_1, b_2, c_3; \mathbf{z})}{F_K(a_1, a_2, b_1 + 1, b_2; a_1, b_2, c_3 + 1; \mathbf{z})} \tag{1}$$

has a formal branched continued fraction of the form

$$1 - z_1 - \cfrac{u_1 z_3}{1 - z_2 - \cfrac{u_2 z_3}{1 - z_1 - \cfrac{u_3 z_3}{1 - z_2 - \cfrac{u_4 z_3}{1 - \ddots}}}}, \tag{2}$$

where, for $k \geq 1$,

$$u_{2k-1} = \frac{(a_2 + k - 1)(c_3 + k - 1 - b_1)}{(c_3 + 2k - 2)(c_3 + 2k - 1)}, \quad u_{2k} = \frac{(b_1 + k)(c_3 + k - a_2)}{(c_3 + 2k - 1)(c_3 + 2k)}. \tag{3}$$

The following theorem is true (see, [2]).

Theorem 2. *Let a_2, b_1, and c_3 be complex constants which satisfy the conditions*

$$|u_k| + \mathrm{Re}(u_k) \leq p\,q(1-q), \quad k \geq 1,$$

where u_k, $k \geq 1$, are defined by (3), $c_3 \notin \{0, -1, -2, \ldots\}$, p is a positive number, $0 < q < 1$. Then the branched continued fraction (2) converges uniformly on every compact subset of the domain

$$\Omega_{p,q}^{u,r} = \Omega_{p,q} \cup \Omega^{u,r}, \tag{4}$$

where

$$\Omega_{p,q} = \left\{ \mathbf{z} \in \mathbb{C}^3 :\ z_k \neq \left[\frac{q}{2}, +\infty\right),\ k = 1, 2,\ |z_3| < \frac{1 + \cos(\arg(z_3))}{2p} \right\}$$

and

$$\Omega^{u,r} = \left\{ \mathbf{z} \in \mathbb{C}^3 :\ |z_k| < \frac{1-r}{2},\ k = 1, 2,\ |z_3| < \frac{r(1-r)}{2u} \right\},$$

where

$$u = \max_{k \in \mathbb{N}} \{|u_k|\}, \quad 0 < r < 1,$$

to a function $f(\mathbf{z})$ holomorphic in $\Omega_{p,q}^{u,r}$, and $f(\mathbf{z})$ is an analytic continuation of the function (1) in (4).

An application of Theorem 2 is the following.

Theorem 3. *Let a_2, b_1, and c_3 be real constants such that*

$$-u \leq u_k < 0, \quad k \geq 1,$$

where u is a positive number, u_k, $k \geq 1$, are defined by (3). Then the branched continued fraction (2) converges uniformly on every compact subset of the domain

$$\Omega_u = \left\{ \mathbf{z} \in \mathbb{C}^3 :\ z_k \notin \left[\frac{1}{2}, +\infty\right),\ k = 1, 2,\ z_3 \notin \left(-\infty, -\frac{1}{8u}\right] \right\} \tag{5}$$

to function $f(\mathbf{z})$ holomorphic in Ω_u, and $f(\mathbf{z})$ is an analytic continuation of the function (1) in (5).

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On some semi-convergences with applications on Korovkin-type theorems

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This work focuses on dealing with the Korovkin-type theorems that are contingent upon the semi-types of "exhaustiveness" and "almost uniform convergence". Since it is known that the convergence types mentioned above are between pointwise and uniform convergence, it will be noticed that the circumstances can be mitigated in the classical Korovkin's Theorem.

Let (X, d) and (Y, ρ) be metric spaces, (f_n) be a sequence of functions and f be a function from X to Y. Let us give the definitions of exhaustiveness, semi-exhaustiveness and semi-uniform convergence below.

Definition 1. *[1] The sequence (f_n) is called exhausitive at $x_0 \in X$, if for every $\varepsilon > 0$ there exists $\delta > 0$ and $n_0 \in \mathbb{N}$ such that for all $x \in B_d(x_0, \delta)$ and all $n \geq n_0$ we have that $\rho(f_n(x), f_n(x_0)) < \varepsilon$.*

Definition 2. *[2] The sequence (f_n) is called semi-exhausitive at $x_0 \in X$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $n \in \mathbb{N}$ there exists $m \in \mathbb{N}(m > n)$ such that for all $x \in B_d(x_0, \delta)$ we have that $\rho(f_m(x), f_m(x_0)) < \varepsilon$.*

Definition 3. *[2] The sequence (f_n) is called semi-uniformly convergent to a function f at x_0 if*

1. $f_n(x_0) \to f(x_0)$

2. *For every* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that for all* $n \in \mathbb{N}$ *there exists* $m \in \mathbb{N}(m > n)$ *such that for all* $x \in B(x_0, \delta)$ *implies* $\rho\left(f_m(x), f(x)\right) < \varepsilon$

Theorem 1. *[3] Let* (L_n) *be a sequence of positive linear operators on* $C(X)$ *and* $x_0 \in X$. *If* $L_n(e_0) \xrightarrow{a.u.} e_0$ *and* $L_n(\rho_{x_0}^r)$ *is almost uniformly converges to* 0 *at* x_0 *for some* $r > 0$, *then* $L_n(f) \xrightarrow{a.u.} f$ *at* x_0 *for all* $f \in C_b(X)$.

Theorem 2. *[3] Let* (L_n) *be a sequence of positive linear operators on* $C_b(X)$ *and* $x_0 \in X$. *If the sequence* $(L_n(e_0))$ *densely semi-uniformly convergent to* e_0 *and the sequence* $L_n(\rho_{x_0}^r)$ *densely semi-uniformly convergent to* 0, *for some* $r > 0$, *at* x_0, *then* $L_n(f)$ *densely semi-uniformly convergent to* f *at* x_0 *for all* $f \in C_b(X)$.

Theorem 3. *[3] Let* (L_n) *be a sequence of positive linear operators on* $C(X)$. *If* $(L_n(e_0))$ *is semi-exhaustive and bounded at* $x_0 \in X$, *then* $(L_n(f))$ *is semi-exhaustive at* x_0 *for all* $f \in C(X)$.

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Quasilinear P.D.E.s, interpolation spaces and Hölderian mappings

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As in the work of Tartar [1], we develop here some new results on nonlinear interpolation of α-Hölderian mappings between normed spaces, by studying the action of the mappings on K-functionals and between interpolation spaces with logarithm functions. We apply these results to obtain some regularity results on the gradient of the solutions to quasilinear equations of the form

$$-\mathrm{div}(\widehat{a}(\nabla u)) + V(u) = f,$$

where V is a nonlinear potential and f belongs to non-standard spaces like Lorentz-Zygmund spaces. We show several results; for instance, that the mapping $\mathcal{T}: \ \mathcal{T}f = \nabla u$ is locally or globally α-Hölderian under suitable values of α and appropriate hypotheses on V and $\widehat{a}$.

The talk is based on the paper [2].

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Extension of the function $f \in W_p^{(r)}(G;s)$ beyond the domain $G \in E_n$ with class preservation

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We consider the space $W_p^{(r)}(G;s)$ of functions $f = f(x)$ of the points $x = (x_1; x_2; ...; x_s) \in E_n$ of many groups of variables $x_k = (x_{k1}; x_{k2}; ...; x_{k,n_k}) \in E_{nk}$ $(n = n_1 + n_2 + ... + n_k)$, satisfying the "$\sigma = (\sigma_1 + \sigma_2, ..., \sigma_s)$ semi-horn condition".

By the method of integral representation based on a new representation of functions, we construct the function

$$\tilde{f} = \tilde{f}(x), \tag{1}$$

determined on the whole E_n, coinciding in the domain $G \in E_n$ with the corresponding

$$f \in W_p^{(r)}(G;s) \tag{2}$$

If only the geometry of the domain $G \in E_n$ corresponds to the smoothness index

$$\sigma = \frac{1}{r} = \left(\frac{1}{r}; ...; \frac{1}{r_s}\right), \sigma_k = \frac{1}{r_k} = \left(\frac{1}{r_{k,1}}, ...; \frac{1}{r_{k,n_k}}\right), \quad k = 1, 2, ..., s.$$

of the space (2)

The integral inequalities

$$\left\|\tilde{f}\right\|_{W_p^{(r)}(E_n;s)} \le C \left\|f\right\|_{W_p^{(r)}(G;s)} \tag{3}$$

of the inclusion type

$$W_p^{(r)}(E_n;s) \in W_p^{(r)}(G;s) \tag{4}$$

are proved.

The obtained results generalize the corresponding results for Sobolev-Slabodetskiy space $W_p^{(r)}(G)$ for $s = 1$, and in the case $s = n$ for the Nikolski spaces $S_p^{(r)}W(G)$ are given in the papers of I.M.Nikolski, O.V.Besov, V.R.Il'in A.J.Jabrailov and others (see [1,2])

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Approximation theorems in Weighted Morrey spaces

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Let $\mathbb{T} := [0, 2\pi]$. For a given weight function $\omega : \mathbb{T} \to (0, \infty)$ the weighted Morrey space $\mathcal{M}_{\omega}^{p,\lambda}(\mathbb{T})$, $0 \le \lambda \le 1$ and $1 \le p < \infty$, is defined as the set of all functions $f \in L_{loc}^{p}(\mathbb{T})$ for which

$$\|f\|_{p,\lambda,\omega} := \left\{ \sup_{I \subset (0,2\pi)} \frac{1}{\omega(I)^{\lambda}} \int_{I} |f(x)|^{p} \omega(x)\, dx \right\}^{1/p} < \infty,$$

where $\omega(I) := \int_{I} \omega(x)\, dx$ and supremum is taken over all subintervals $I \subset \mathbb{T}$. We will assume that the weight functions ω are from the well known Muckenhoupt class $A_p(\mathbb{T})$. Note that $\mathcal{M}_{\omega}^{p,\lambda}(\mathbb{T})$, becomes a Banach spaces equipped with the norm $\|\cdot\|_{p,\lambda,\omega}$. By $L_{\omega}^{p,\lambda}(\mathbb{T})$ we will denote the closure of trigonometric polynomials in $\mathcal{M}^{p,\lambda}(\mathbb{T})$.

In this talk we discuss approximation problems in the spaces $L_{\omega}^{p,\lambda}(\mathbb{T})$, $0 \le \lambda < 1$, $1 < p < \infty$, when $\omega \in A_p(\mathbb{T})$. We define modulus of smoothness in these spaces and in term of this modulus we obtain some direct and inverse theorems of approximation theory by trigonometric polynomials. Moreover, some Lipschitz subclasses of weighted Morrey spaces are defined and theirs constructive characterizations are obtained.

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Approximation by trigonometric polynomials in weighted grand Lorentz spaces

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In the present work, the relationship between the best approximation of the function and the best approximation of the derivatives of the function in the weighted grand Lorentz spaces $\widetilde{L}_{\omega}^{p)s,\varphi}(\mathbb{T})$ (see. [1], [2], [3] and [4]) has been investigated. We study the problem of the best approximation of the functions in certain subspace of the weighted grand Lorentz spaces. Also, we prove an inverse theorem of approximation theory in weighted grand Lorentz spaces.

Let $1 < p < \infty$. By Φ_p we denote the set of positive measurable functions φ defined on $(0, p-1)$ when are nondecreasing, bounded with a condition $\lim\limits_{x\to 0^+} \varphi(x) = 0$.

Let $1 < p < \infty$, $p' = \frac{p}{p-1}$ and let ω be a weight function on $\mathbb{T}$. ω is said to satisfy *Muckenhoupt's A_p-condition* on $\mathbb{T}$ if

$$\sup_J \left(\frac{1}{|J|} \int\limits_J \omega(t)\, dt \right) \left(\frac{1}{|J|} \int\limits_J \omega^{1-p'}(t)\, dt \right)^{p-1} < \infty ,$$

where J is any subinterval of $\mathbb{T}$ and $|J|$ denotes its length [5].

Let $1 < p < \infty$ and let $\omega \in A_p(\mathbb{T})$. For $f \in \widetilde{L}_{\omega}^{p),s,\varphi}(\mathbb{T})$ we define the *modulus of smoothness* $\Omega(f,\cdot)_{L_{\omega}^{p),s,\varphi}(\mathbb{T})} : [0,\infty) \longrightarrow [0,\infty)$ of f by [3]

$$\Omega(f,\cdot)_{L_{\omega}^{p),s,\varphi}(\mathbb{T})} = \sup_{|0<h|\leq\delta} \left\| \frac{1}{2h} \int_{x-h}^{x+h} f(t)\,dt - f(x) \right\|_{L_{\omega}^{p),s,\varphi}(\mathbb{T})} \delta > 0 \ ,$$

The best approximation of $f \in \widetilde{L}_{\omega}^{p),s,\varphi}(\mathbb{T})$ in the class Π_n of trigonometric polynomials of degree not exceeding n is defined by

$$E_n(f)_{L_{\omega}^{p),s,\varphi}(\mathbb{T})} := \inf \left\{ \|f - T_n\|_{L_{\omega}^{p),s,\varphi}(\mathbb{T})} : T_n \in \Pi_n \right\}.$$

Our main results are the following.

Theorem 1. *Let $1 < p < \infty$, $\varphi \in \Phi_p$, $\omega \in A_p(T)$ and T_n be the best approximation by trigonometric polynomials to f in the space $\widetilde{L}_{\omega}^{p),s,\varphi}(\mathbb{T})$. If $f \in \widetilde{L}_{\omega}^{p),s,\varphi}(\mathbb{T})$ satisfies for some natural r*

$$\sum_{m=1}^{\infty} m^{r-1} E_m(f)_{L_{\omega}^{p),s,\varphi}(\mathbb{T})} < \infty$$

then the inequality

$$\left\| f^{(r)} - T_n^{(r)} \right\|_{L_{\omega}^{p),s,\varphi}(\mathbb{T})} \leq c \left\{ n^r E_n(f)_{L_{\omega}^{p),s,\varphi}(\mathbb{T})} + \left(\sum_{\mu=n+1}^{\infty} \mu^{r-1} E_\mu(f)_{L_{\omega}^{p),s,\varphi}(\mathbb{T})} \right) \right\}.$$

holds.

Theorem 2. *Let $1 < p < \infty$, $\varphi \in \Phi_p$ and $\omega \in A_p(T)$. Let $f \in \widetilde{L}_{\omega}^{p),s,\varphi}(\mathbb{T})$ and*

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

is its Fourier series and let

$$\sum_{n=1}^{\infty} E_n(f)_{L_{\omega}^{p),s,\varphi}(\mathbb{T})} n^{\alpha-1} < \infty,$$

where $\alpha \in \mathbb{R}$. *Then the series*

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} n^{\alpha} \left(a_n \cos nx + b_n \sin nx\right)$$

is the Fourier series of the some function $\widetilde{f} \in \widetilde{L}_{\omega}^{p),s,\varphi}(\mathbb{T})$ *and for every* $\widetilde{f} \in \widetilde{L}_{\omega}^{p),s,\varphi}(\mathbb{T})$ *the estimates*

$$\begin{aligned} & E_n(\widetilde{f})_{L_{\omega}^{p),s,\varphi}(\mathbb{T})} \\ \leq\ & c_1 \left[E_n(f)_{L_{\omega}^{p),s,\varphi}(\mathbb{T})} n^{\alpha} + \sum_{k=n+1}^{\infty} E_k(f)_{L_{\omega}^{p),s,\varphi}(\mathbb{T})} k^{\alpha-1} \right], n = 1, 2, ... \end{aligned}$$

$$E_0(\widetilde{f})_{L_{\omega}^{p),s,\varphi}(\mathbb{T})} \leq c_2 \left[E_0(f)_{L_{\omega}^{p),s,\varphi}(\mathbb{T})} + \sum_{k=1}^{\infty} E_k(f)_{L_{\omega}^{p),s,\varphi}(\mathbb{T})} k^{\alpha-1} \right]$$

hold with the constants c_1, $c_2 > 0$, *nondependent on* f *and* n.

Corollary 1. *Under the conditions of Theorem 2 the estimate*

$$\Omega\left(\widetilde{f}, \delta\right)_{L_{\omega}^{p),s,\varphi}(\mathbb{T})} \leq c_3 \left\{ \frac{1}{n^2} \sum_{\nu=0}^{n-1} (\nu+1)^{\alpha+1} E_{\nu}(f)_{L_{\omega}^{p),s,\varphi}(\mathbb{T})} + \sum_{s=n}^{\infty} s^{\alpha-1} E_s(f)_{L_{\omega}^{p),s,\varphi}(\mathbb{T})} \right\}$$

holds with a constant $c_3 > 0$, *depending on* α *and* r.

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On the error of approximation by RBF neural networks with two hidden nodes

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The set of RBF neural networks considered in this paper consists of the following functions

$$\sum_{i=1}^{m} w_i g\left(\frac{\|\mathbf{x}-\mathbf{c_i}\|}{\sigma_i}-\theta_i\right). \tag{1}$$

Here $m \in \mathbb{N}$ is the number of nodes in the hidden layer, $(w_1, ..., w_m) \in \mathbb{R}^m$ is the vector of weights, $\mathbf{x} \in \mathbb{R}^d$ is an input vector, $\mathbf{c_i} \in \mathbb{R}^d$ and $\sigma_i \in \mathbb{R}$ are the centroids and smoothing factor (or width) of the i-th node, $1 \leq i \leq m$, respectively, $\theta_i \in \mathbb{R}$ are thresholds and $g : \mathbb{R} \to \mathbb{R}$ is the so-called activation function.

Let $g(x)$ be a continuous activation function on $\mathbb{R}$, consider the approximation of the continuous function $f(\mathbf{x}) = f(x_1, ..., x_d)$ on a compact subset $Q \subset \mathbb{R}^d$ using a set of radial basis function (RBF) neural networks:

$$\mathcal{G} = \mathcal{G}(g, \mathbf{c_1}, \mathbf{c_2}) = \left\{\sum_{i=1}^{m} w_i g\left(\frac{\|\mathbf{x}-\mathbf{c_i}\|}{\sigma_i}-\theta_i\right) : w_i, \sigma_i, \theta_i \in \mathbb{R};\ \mathbf{c_i} = \mathbf{c_1} \ \ or \ \ \mathbf{c_i} = \mathbf{c_2}\right\}. \tag{2}$$

In equation (2), $\mathbf{c}_1$ and $\mathbf{c}_2$ are fixed center points of the radial basis functions used in the RBF neural network. On the other hand, the numbers w_i, σ_i, and θ_i are variables that can vary and need to be determined during the process.

The approximation error is defined as follows:

$$E(f) = E(f, \mathcal{G}) \stackrel{def}{=} \inf_{u \in \mathcal{G}} \|f - u\|,$$

where

$$\|f - u\| = \max_{\mathbf{x} \in Q} |f(\mathbf{x}) - u(\mathbf{x})|.$$

Assume $Q \subset \mathbb{R}^d$ and $\mathbf{c}_1, \mathbf{c}_2 \in \mathbb{R}^d$ are fixed center points.

Definition 2.1. *A finite or infinite ordered set* $p = (\mathbf{p}_1, \mathbf{p}_2, ...) \subset Q$ *with* $\mathbf{p}_i \neq \mathbf{p}_{i+1}$, *and either* $\|\mathbf{p}_1 - \mathbf{c}_1\| = \|\mathbf{p}_2 - \mathbf{c}_1\|, \|\mathbf{p}_2 - \mathbf{c}_2\| = \|\mathbf{p}_3 - \mathbf{c}_2\|$, $\|\mathbf{p}_3 - \mathbf{c}_1\| = \|\mathbf{p}_4 - \mathbf{c}_1\|, ...$ *or* $\|\mathbf{p}_1 - \mathbf{c}_2\| = \|\mathbf{p}_2 - \mathbf{c}_2\|, \|\mathbf{p}_2 - \mathbf{c}_1\| = \|\mathbf{p}_3 - \mathbf{c}_1\|$, $\|\mathbf{p}_3 - \mathbf{c}_2\| = \|\mathbf{p}_4 - \mathbf{c}_2\|, ...$ *is called a path with respect to the centers* $\mathbf{c}_1$ *and* $\mathbf{c}_2$.

A finite path $(\mathbf{p}_1, \mathbf{p}_2, ..., \mathbf{p}_{2n})$ is said to be closed if $(\mathbf{p}_1, \mathbf{p}_2, ..., \mathbf{p}_{2n}, \mathbf{p}_1)$ is also a path.

Let's consider also the following class of functions denoted as $\mathcal{D}$.

$$\mathcal{D} = \{r_1(\|\mathbf{x} - \mathbf{c}_1\|) + r_2(\|\mathbf{x} - \mathbf{c}_2\|) : \ r_i \in C(\mathbb{R}), \ i = 1, 2\}.$$

Note that, in the definition of $\mathcal{G} = \mathcal{G}(g, \mathbf{c_1}, \mathbf{c_2})$ each term $w_i g\left(\frac{\|\mathbf{x}-\mathbf{c_i}\|}{\sigma_i} - \theta_i\right)$ can be interpreted as a function $h(\mathbf{x} - \mathbf{c_i})$ with $\mathbf{c_i} = \mathbf{c_1}$ or $\mathbf{c_i} = \mathbf{c_2}$. The function h depends on the parameters w_i, σ_i and θ_i. It is evident that an element $v \in \mathcal{G} = \mathcal{G}(g, \mathbf{c_1}, \mathbf{c_2})$ also belongs to the class $\mathcal{D}$. In other words, $\mathcal{G} = \mathcal{G}(g, \mathbf{c_1}, \mathbf{c_2})$ is a subset of the class $\mathcal{D}$.

For each closed path $p = (\mathbf{p}_1, \mathbf{p}_2, ..., \mathbf{p}_{2n})$ let us consider the following functional:

$$G_p(f) = \frac{1}{2n} \sum_{k=1}^{2n} (-1)^{k+1} f(\mathbf{p}_k).$$

This functional is associated with the closed path p and exhibits the following obvious properties:

(a) If $r \in \mathcal{D}$, then $G_p(r) = 0$.

(b) $\|G_p\| \leq 1$ and if $p_i \neq p_j$ for all $i \neq j$, $1 \leq i, j \leq 2n$, then $\|G_p\| = 1$.

The following lemma is valid.

Lemma 1. *Let a compact set Q have closed paths. Then*

$$\sup_{p\subset Q}|G_p(f)|\leq \inf_{u\in\mathcal{G}}\|f-u\|\,, \tag{3}$$

where the sup is taken over all closed paths.

The images of the distance functions $\|\mathbf{x}-\mathbf{c}_1\|$ and $\|\mathbf{x}-\mathbf{c}_2\|$ on the compact set Q are denoted by X_1 and X_2, respectively. For any function $h\in C(Q)$, consider the real functions

$$\begin{aligned} s_1(a) &= \max_{\substack{\mathbf{x}\in Q\\ \|\mathbf{x}-\mathbf{c}_1\|=a}} h(x),\ s_2(a)=\min_{\substack{\mathbf{x}\in Q\\ \|\mathbf{x}-\mathbf{c}_1\|=a}} h(x), a\in X_1,\\ g_1(b) &= \max_{\substack{\mathbf{x}\in Q\\ \|\mathbf{x}-\mathbf{c}_2\|=b}} h(x),\ g_2(b)=\min_{\substack{\mathbf{x}\in Q\\ \|\mathbf{x}-\mathbf{c}_2\|=b}} h(x),\ b\in X_2. \end{aligned}$$

When are these functions continuous on the appropriate sets X_1 and X_2? The following lemma, answers this question.

Lemma 2. *Let $Q\subset\mathbb{R}^d$ be a compact set. Then the functions s_1 and s_2 are continuous on X_1 (g_1 and g_2 are continuous on X_2) for any $h\in C(Q)$ if the following condition, which we call the condition A, holds:*

(A) for any two points $\mathbf{x}$ and $\mathbf{y}$ in Q with $\|\mathbf{x}-\mathbf{c}_1\|=\|\mathbf{y}-\mathbf{c}_1\|$ ($\|\mathbf{x}-\mathbf{c}_2\|=\|\mathbf{y}-\mathbf{c}_2\|$) and any sequence $\{\mathbf{x}_n\}_{n=1}^{\infty}$ tending to $\mathbf{x}$, there exists a sequence $\{\mathbf{y}_n\}_{n=1}^{\infty}$ tending to $\mathbf{y}$ such that $\|\mathbf{x}_n-\mathbf{c}_1\|=\|\mathbf{y}_n-\mathbf{c}_1\|$ ($\|\mathbf{x}_n-\mathbf{c}_2\|=\|\mathbf{y}_n-\mathbf{c}_2\|$) for all $n=1,2,...$

The following theorem is true.

Theorem 1. *Let $Q\subset\mathbb{R}^d$ be a compact set and $f\in C(Q)$. Suppose the following conditions hold.*

1) f has a best approximation in $\mathcal{D}$;

2) The above condition (A) holds;

3) there exists a positive integer N_0 such that any path $(\mathbf{p}_1,...,\mathbf{p}_n)\subset Q$, $n>N_0$, or a subpath of it can be made closed by adding not more than N_0 points of Q.

Then for any continuous nonpolynomial activation function $g:\mathbb{R}\to\mathbb{R}$ the approximation error by RBF neural networks with two hidden nodes $\mathcal{G}=$

$\mathcal{G}(g, \mathbf{c_1}, \mathbf{c_2})$ can be computed by the formula

$$E(f, \mathcal{G}) = \sup_{p \subset Q} |G_p(f)|,$$

where the sup is taken over all closed paths.

One Bernstein-type relation in rational approximation theory

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Consider three sets in the complex plane: T, D_+, and D_-, defined as the circle $|z| = 1$, the circle $|z| < 1$, and the domain $|z| > 1$ respectively. For $0 < p \leqslant \infty$ we denote by $L_p(D_+)$ the Lebesgue space of complex functions on D_+ with respect to the flat Lebesgue measure with the usual quasi-norm $\|f\|_{L_p(D_+)}$. By $\mathcal{R}_n$, $n = 0,\ 1,\ 2,\ \ldots$, we denote the set rational functions of degree at most n with poles only in D_-. It is known that $r_n \in \mathcal{R}_n \bigcap L_\infty(D_+)$ satisfies the estimate

$$\|r_n'\|_{L_2(D_+)} \leqslant \sqrt{\pi\, n}\ \|r_n\|_{L_\infty(D_+)},$$

received by E.P. Dolzhenko from geometrical considerations. Various generalizations of this result exist in rational approximation theory, including extensions to Lebesgue spaces $L_p(D_+)$ with respect to flat measures, higher derivatives, and fractional order derivatives, along with corresponding inverse theorems ([3],[4]).

For $\alpha \in \mathbb{R}$ and $0 < p \leqslant \infty,\ 0 < q \leqslant \infty$, denote by B_q^α the Hardy—Besov space(see, for example, [1],[2]). Namely, $f \in B_q^\alpha$ if for some $\beta > \alpha$ the function $(1-|z|^2)^{\beta-\alpha-\frac{1}{q}} \cdot \left(J^\beta f\right)(z)$ belongs to $L_q(D_+)$, where $J^\beta f$ is the Weyl derivative of the function f. Quasi-norm (norm at $1 \leqslant q \leqslant \infty$) in the space B_q^α is defined as follows

$$\|f\|_{B_q^\alpha} = \left\| \left(1-|z|^2\right)^{\beta-\alpha-\frac{1}{q}} \cdot \left(J^\beta f\right)(z) \right\|_{L_q(D_+)} =$$

$$= \left(\int\limits_{D_+} \left| \left(1 - |z|^2\right)^{\beta - \alpha - \frac{1}{q}} \cdot \left(J^{\beta} f\right)(z) \right|^q dm_2(z) \right)^{\frac{1}{q}} < \infty \ .$$

It is well known that the definition of the space B_q^α does not depend on β: for different β the corresponding quasi-norms are equivalent. For the sake of convenience, as a rule, we assume $\beta = \alpha + 1$.

Theorem 1.[3] *Let $r \in \mathcal{R}_n$, $\alpha > 0$, $p > 2$ and $1/q = \alpha + 2/p$. Then*

$$\|r\|_{B_q^\alpha} \leqslant c(\alpha, p) n^{\alpha + 1/p} \|r\|_{L_p(D_+)} \,,$$

where $c > 0$ and depends only on α and p.

We further assume that a rational function r of degree $n + m$ has no poles on T, and n poles lie in D_+ and m — in D_-. Then $r(z) = r_+(z) + r_-(1/z)$, where r_+ and r_- are rational functions of degree n and m, respectively, with poles only at D_-. From Theorem 1 we immediately obtain.

Theorem 2. *Let $\alpha > 0$, $p > 2$ and $1/q = \alpha + 2/p$. Then*

$$\|r_+\|_{B_q^\alpha} \leqslant c(\alpha, p) n^{\alpha + 1/p} \|r\|_{L_p(D_+)},$$

$$\|r_-\|_{B_q^\alpha} \leqslant c(\alpha, p) m^{\alpha + 1/p} \|r\|_{L_p(D_+)} \,,$$

where $c > 0$ and depends only on α and p.

By employing the standard Bernstein method, one can obtain corresponding inverse theorems, where the technical apparatus involves the relations given here for derivatives of rational functions.

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On embedding theorems in small Besov spaces

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In this abstract we introduce a small Besov space $B^l_{(p,(\theta}(G)$, and with help of method integral representation we study differential properties of functions from these spaces.

Let $G \subset R^n$ ($|G| < \infty$) satisfy flexible λ $(\lambda \in (0, \infty)$)-horn condition, m_i $(i = 1, 2, \ldots, n)$ are positive integers and k_i $(i = 1, 2, \ldots, n)$ are non-negative integers, h_0 is fixed positive number, $1 < p < \infty$, $1 < \theta < \infty$.

Definition. *Small of Besov spaces $B^l_{(p,(\theta}(G)$, we denote the spaces of all functions $f \in L_1^{loc}(G)$ with the finite norm* $(m_i > l_i - k_i > 0;\ i = 1, 2, \ldots, n)$

$$\|f\|_{B^l_{(p,(\theta}(G)} = \|f\|_{L_{(p}(G)} + \sum_{i=1}^n \|f\|_{\mathcal{L}^l_{(p,(\theta}(G)},$$

where

$$\|f\|_{L_{(p}(G)} = \inf_{0<\varepsilon_1<p-1} \left(\frac{\varepsilon^{-\frac{p-\varepsilon_1}{(p-\varepsilon_1)'}}}{|G|} \int_G |f(x)\ |^{p-\varepsilon_1} dx \right)^{\frac{1}{p-\varepsilon_1}}, \ \ (p-\varepsilon_1)' = \frac{p-\varepsilon_1}{p-\varepsilon_1-1},$$

$$\|f\|_{\mathcal{L}^{l_i}_{(p,(\theta}(G)} = \sup_{0<\varepsilon_2<\theta-1} \left(\frac{\varepsilon_2}{h_0} \right)^{\frac{1}{\theta-\varepsilon_2}} \times$$

$$\times \left\{ \int_0^{h_0} \left[\frac{\|\Delta_i^{m_i}\left(h^{\lambda_i}, G_{h^\lambda}\right) D_i^{k_i} f\|_{(p}}{h^{\lambda_i(l_i-k_i)}} \right]^{\theta-\varepsilon_2} \frac{dh}{h} \right\}^{\frac{1}{\theta-\varepsilon_2}},$$

$$\Delta_i^{m_i}\left(h^{\lambda_i}, G_{h^\lambda}\right) f\left(x\right) =$$
$$= \begin{cases} \Delta_i^{m_i}\left(h^{\lambda_i}\right) f\left(x\right) & \text{for} \quad \left[x, x + m_i h^{\lambda_i} e_i\right] \subset G_{h^\lambda}, \\ 0 & \text{for} \quad \left[x, x + m_i h^{\lambda_i} e_i\right] \not\subset G_{h^\lambda}, \end{cases}$$
$$\Delta_i^{m_i}\left(h^{\lambda_i}\right) f\left(x\right) = \sum_{j=0}^{m_i} \left(-1\right)^{m_i - j} c_{m_i}^{j} f\left(x + j h^{\lambda_i} e_i\right), \quad e_i = \Bigg(0, \underbrace{\ldots}_{i-1}, 0, 1, 0, \ldots, 0\Bigg).$$

In the case $\theta = \infty$ this space coincides with the small Nikolski spaces.

The following embedding theorems:

1. $D^\nu : B^l_{(p,(\theta}\left(G\right) \hookrightarrow L_{q-\varepsilon_1}\left(G\right)$ for all $0 < \varepsilon_1 < p - 1$;
2. $D^\nu : B^l_{(p,(\theta}\left(G\right) \hookrightarrow B^{l^1}_{q-\varepsilon_1, \theta-\varepsilon_2}\left(G\right)$ for all $0 < \varepsilon_2 < \theta - 1, l^1 \in (0, \infty)$.

Matrix characterization of asymptotically deferred I-equivalent sequences

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In 1932 Agnew [1] presented the concept of deferred Cesáro means. Also, the notion of $I-$ convergence was introduced by Kostyrko et al. in [2]. By considering their notions, the main goal of this paper is to study the concept of deferred I- equivalence of two nonnegative sequences. Additionally, important conditions for a matrix $A = (a_{jk})$ to be asymptotic $I-$regular are presented.

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mcv-polar sets

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We will introduce the concept of mcv-polar sets and show a number of their properties. As we see below, the situation here differs significantly from the known classes of polar and pluripolar sets. For example, in the case $m = 1$ there is no polar set at all.

1. m-convex functions $(m - cv)$ are the real analogue in $\mathbb{R}^n$ of strongly m-subharmonic (sh_m) functions in a complex space $\mathbb{C}^n$: function $u(x) \in C^2(D)$ is said to be m-convex if Hessians

$$H^s(u) = H^s(\lambda) = \sum_{1 \le j_1 < \ldots < j_s \le n} \lambda_{j\,1} \ldots \lambda_{j_s}$$

of degree s of the eigenvalue vector $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n)$ satisfy the conditions

$$H^s\left(\lambda(x)\right) \ge 0, \quad \forall x \in D, \;\; s = 1, ..., n - m + 1.$$

Thus, when $m = 1$ they are geometrically convex functions (see. [1-3]) and when $m = n$ subharmonic functions in the domain $D \subset \mathbb{R}^n$ (see. [4]).

Similar functions are discussed in the series works by N.Trudinger, N. Ivochkin, X. Wang and others (see [8-14]).

Using the known connection between $m - cv$ functions and well-known $sh_m(D)$ ones, we can define $m - cv$ functions in the class integrable functions: consider the natural embedding of a real space $\mathbb{R}^n_x$ into a complex space $\mathbb{C}^n_z$, $z = x + iy$. For a function $u(x)$, $x \in D \subset \mathbb{R}^n_x$, we set $u^c(z) = u^c(x + iy) = u(x)$, $z \in D \times \mathbb{R}^n_y \subset \mathbb{C}^n_z$.

Definition 1. A function $u(x) \in L^1_{loc}(D)$ is called m-convex function in the domain $D \subset \mathbb{R}^n_x$, i.e. $u(x) \in m - cv(D)$, if it is upper semicontinuous and $u^c(z) \in sh_m(D \times \mathbb{R}^n_y)$.

2. mcv**-polar sets**

Definition 2. By analogy with polar sets, $E \subset D \subset \mathbb{R}^n$ is called *mcv*-polar set in D, if there exists a function $u(x) \in m - cv(D)$, $u(x) \not\equiv -\infty$, such that $u\,|_E = -\infty$.

From the embedding $m - cv(D) \subset sh(D)$ it follows that every *mcv*-polar set is polar. In particular, for a *mcv*-polar set E it is true $H_{n-2+\varepsilon}(E) = 0$ $\forall \varepsilon > 0$, and therefore the Lebesgue measure of a *mcv*-polar set E is also equal to zero. Using the connection of $m - cv$ and $sh_m(D)$ functions it is very easy to prove

Theorem 1. *The countable union of mcv-polar sets is mcv-polar, i.e. if the sets $E_j \subset D$ are mcv-polar, then $E = \bigcup_{j=1}^{\infty} E_j$ is also mcv-polar.*

Theorem 2. (see also works [1-3]). *Function $u(x) \in 1-cv(D)$, $u(x) \not\equiv -\infty$ is continuous everywhere on D.*

Progress of the proof. Let us use the following statements: $u(x) \in 1-cv(D)$, if and only if when $u^c(z) \in sh_1(D \times \mathbb{R}^n_y) \cup psh(D \times \mathbb{R}^n_y)$ and the well-known analogue of Cartan's theorem about any *psh* function $u^c(z)$ is continuous almost everywhere in the $\mathcal{P}$-capacity, that for any $\varepsilon > 0$ there exists an open set $U_\varepsilon \subset D \times \mathbb{R}^n_y :\ C(U_\varepsilon) < \varepsilon$ and $u^c(z)$ will be continuous on $\left[D \times \mathbb{R}^n_y\right] \setminus U_\varepsilon$. If $u(x) \in 1 - cv(D)$ is discontinuous at the point x^0, then the function $u^c(z)$ will be discontinuous on the real plane $\{x^0\} \times \mathbb{R}^n_y$. But this plane has positive $\mathcal{P}$-capacity, which contradicts Cartan's theorem.

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On some shape preserving properties of Bernstein operators

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One of the most important problems in approximation theory in mathematical analysis is the determination of sequences of polynomials that converge to functions and have the same geometric properties. This type of approximation is called the shape-preserving approximation. These types of problems are usually handled depending on the convexity of the functions, the degree of smoothness depending on the order of differentiability, or whether it satisfies a functional equation. The problem addressed in this paper belongs to the third class. A quadratic bivariate algebraic equation denotes geometrically some well-known shapes such as circle, ellipse, hyperbola and parabola. Such equations are known as conic equations. In this study, it is investigated whether conic equations transform into a conic equation under bivariate Bernstein polynomials, and if so, which conic equation it transforms into. Also, we present some preservation properties of Bernstein Operators of one and two variables such as $\mathbb{B}$-convexity and $\mathbb{B}$-concavity of functions defined in [1-2]. We give examples which Bernstein polynomials of two variables do not preserve convexity properties of these functions. In addition, for these convexities, results are given regarding conditions it will be preserved.

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HARMONIC AND NON-HARMONIC ANALYSIS

The Weighted Grand Lebesgue class of harmonic functions and the Dirichlet problem

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Consider the following Dirichlet problem

$$\left.\begin{array}{l}\Delta u = 0\,,\ in\ \omega, \\ u/_{\partial\omega} = f\,,\ on\ \partial\omega,\end{array}\right\} \qquad (1)$$

where $f \in \partial\omega \to R = (-\infty, \infty)$ is a given function. There are some kinds of statements (consider e.g. the work [1]) regarding the problem (1). Problem (1) is a model case for studying the solvability of the Dirichlet problem for a second-order elliptic equation. Therefore, the study of the solvability of the problem (1) with respect to various Banach Function Spaces is very important. We will call this problem considered a Hardy-type statement. A Hardy-type statement is a generalization of other statements such as weak solution, strong solution, and classical solution.

In this work, we consider a Hardy-type statement for problem (1). We define a weighted grand Lebesgue class $h_{p);w}$ of harmonic functions in unit ball $\omega = \{z \in C : |z| < 1\}$ on the complex plane C. Some properties of functions of these classes are established when the weight function $w : \partial\omega \to \bar{R}_+ = [0, +\infty]$ satisfy the A_p Muckenhoupt condition. The correct solvability of problem (1) in the class $h_{p);w}$ with nontangential values $f \in L_{p);w}(\partial\omega)$ is proven.

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General solution of homogeneous Riemann problem in banach hardy classes

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This work deals with the rearrangement invariant Banach function space X and Banach Hardy classes generated by this space, which consist of analytic functions inside and outside the unit circle. In these Hardy classes, we consider homogeneous Riemann problems with piecewise continuous coefficients. We constructed the general solution of the homogeneous Riemann problem in Banach Hardy classes.

Consider the homogeneous Riemann problem

$$F^{+}(\tau)-G(\tau)\,F^{-}(\tau)=0,\tau\in\gamma,\quad F^{+}(\cdot)\in H_X^{+};F^{-}(\cdot)\in {}_mH_X^{-},\qquad(1)$$

with complex-valued coefficient $G\left(e^{it}\right)=\left|G\left(e^{it}\right)\right|e^{i\theta(t)},t\in[-\pi,\pi]$. By the solution of the problem (1) we mean a pair of analytic functions $\left(F^{+};F^{-}\right)\in H_X^{+}\times{}_mH_X^{-}$, whose non-tangential boundary values satisfy the equation (1) a.e. on γ. We assume that the coefficient $G(\cdot)$ satisfies the following conditions:

i) $G^{\pm1}(\cdot)\in L_\infty(-\pi,\pi)$;

ii) $\theta(t) = \arg G(e^{it})$ is a piecewise Hölder function on $[-\pi, \pi]$ with the jumps $h_k = \theta(s_k + 0) - \theta(s_k - 0)$, $k = \overline{1, r}$, at the points of discontinuity$\{s_k\}_1^r$: $-\pi < s_1 < \ldots < s_r < \pi$.

Consider the following analytic functions in $C \backslash \gamma$:

$$Z_1(z) \equiv \exp\left\{\frac{1}{4\pi}\int_{-\pi}^{\pi} \log\left|G\left(e^{it}\right)\right| H(t; z)\, dt\right\},$$

$$Z_2(z) \equiv \exp\left\{\frac{i}{4\pi}\int_{-\pi}^{\pi} \theta(t)\, H(t; z)\, dt, z \notin \gamma\right\}.$$

It is absolutely clear that the function $Z_2(\cdot)$ depends on the choice of the argument $\theta(\cdot)$. Let

$$Z_\theta(z) = Z_1(z)\, Z_2(z), z \notin \gamma.$$

$Z_\theta(\cdot)$ will be called a canonical solution of homogeneous problem (1), corresponding to the argument $\theta(\cdot)$.

The following theorem is proved.

Theorem 1. *Let X be a r.i.s. with Boyd indices $0 < \alpha_X \leq \beta_X < 1$. Suppose that the coefficient $G(\cdot)$ of the problem (1) satisfies the conditions i), ii) and $Z_\theta(\cdot)$ is a canonical solution corresponding to argument $\theta(\cdot)$. Let jumps $\{h_k\}_0^r$ of the function $\theta(\cdot)$ ($h_0 = \theta(-\pi) - \theta(\pi)$), satisfy the inequalities*

$$\gamma_{X'} < \frac{h_k}{2\pi} < -\gamma_X, k = \overline{0, r};$$

Then :

α) for $m \geq 0$ the homogeneous problem (1) has a general solution of the form

$$F(z) \equiv Z_\theta(z)\, P_k(z),$$

in the Hardy classes $H_X^+ \times {}_mH_X^-$, where $P_k(z)$ is an arbitrary polynomial of degree $k \leq m$;

β) for $m < 0$ this problem has only a trivial solution, i.e. zero solution in the Hardy classes $H_X^+ \times {}_mH_X^-$.

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Boundedness of Gegenbauer fractional maximal operator generalized Gegenbauer-Morrey spaces

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In this paper Spanne - Guliyev, Adams - Guliyev, Adams - Gunavan and Gunavan- Guliyev types results on the boundedness of G-fractional maximal operator M_G^α on generalized Gegenbauer-Morrey (G-Morrey) spaces were obtained. Moreover, we characterize the boundedness of the k-th order commutator $M_G^{b,\alpha,k}$ of the G-fractional maximal operator on generalized G- Morrey space.

Theorem A. *Let* $1 < p < q < \infty, \quad 0 < \alpha < n$ *and* $1 \le p < n/\alpha$.

(i) If $1 < p < n/\alpha$, *then the condition*

$$\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}$$

is necessary and sufficient for the boundedness I_α *from* $L_p(\mathbb{R}^n)$ *to* $L_q(\mathbb{R}^n)$.

(ii) If $p = 1 < q < \infty$, *then the condition*

$$1 - \frac{1}{q} = \frac{\alpha}{n}$$

is necessary and sufficient for the boundedness I_α *from* $L_1(\mathbb{R}^n)$ *to* $WL_q(\mathbb{R}^n)$.

In [1] for the I_α in the Morrey space, Adams proved the following theorem.

Theorem B. *(Adams [1]) Let* $0 \le \alpha < n$, $0 \le \lambda < n$ *and* $1 \le p < (n-\lambda)/\alpha$

(i) If $1 < p < (n-\lambda)/\alpha$, *then the condition*

$$\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n-\lambda}$$

is necessary and sufficient for the boundedness I_α *from* $\mathcal{M}_{p,\lambda}(\mathbb{R}^n)$ *to* $\mathcal{M}_{q,\lambda}(\mathbb{R}^n)$.

(ii) If $p = 1$, *then the condition*

$$1 - \frac{1}{q} = \frac{\alpha}{n-\lambda}$$

is necessary and sufficient for the boundedness I_α *from* $\mathcal{M}_{1,\lambda}(\mathbb{R}^n)$ *to* $W\mathcal{M}_{q,\lambda}(\mathbb{R}^n)$.

Theorema C. *(Spanne [37]) Let* $0 \le \alpha < n$, $1 < p < n/\alpha$, $0 < \lambda < n - \alpha p$ *and* $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}$.

Then:

(a) By $p > 1$, I_α *is bounded from* $\mathcal{M}_{p,\lambda}(\mathbb{R}^n)$ *to* $\mathcal{M}_{q,\mu}(\mathbb{R}^n)$ *if and only if* $\lambda/p = \mu/q$.

(b) By $p = 1$, I_α *is bounded from* $\mathcal{M}_{p,\lambda}(\mathbb{R}^n)$ *to* $W\mathcal{M}_{q,\mu}(\mathbb{R}^n)$ *if and only if* $\lambda/p = \mu/q$.

Theorem D. *(Komori-Mizuhara [33]) Let* $0 \le \alpha < n$, $1 < p < n/\alpha$, $0 < \lambda < n - \alpha p$ *and* $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n-\lambda}$.

Then the following statements are equivalents:

(a) $b \in BMO\,(\mathbb{R}^n)$.

(b) $[b, I_\alpha]$ *is bounded from* $\mathcal{M}_{p,\lambda}(\mathbb{R}^n)$ *to* $L_{q,\mu}(\mathbb{R}^n)$.

Remark 2.3. *Denote by* Ω_p^γ *a set off all positive measurable functions* ω *on* $\mathbb{R}_+$ *such that for all* $r > 0$

$$\sup_{x\in\mathbb{R}_+} \left\| \frac{\left(sh\frac{r}{2}\right)^{-\frac{\gamma}{p}}}{\omega(x,r)} \right\|_{L_\infty(t,\infty)} < \infty \quad \text{and} \quad \sup_{x\in\mathbb{R}_+} \left\|\omega(x,r)^{-1}\right\|_{L_\infty(0,t)} < \infty,$$

respectively. Lemma 2.1 shows, that it makes sense to consider only functions ω, *from* Ω_p^γ, *which we will assume in what follows.*

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Limits of sequences of operators associated with Walsh system

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Let $f \in L_1(\mathbb{I})$, $\mathbb{I} := [0,1)$ and $S_k(f;x)$ denotes the kth partial sums of the Fourier series with respect to the Walsh system. Assume that $\mathbf{q} := \{q_k : k \geq 0\}$ be a sequence of non-negative numbers. Let us consider a sequence of linear operators defined by

$$T_n^{(\mathbf{q})}(f;x) := \frac{1}{Q_n} \sum_{k=1}^{n} q_{n-k} S_k(f;x), \quad n \in \mathbb{N},$$

where

$$Q_n := \sum_{k=0}^{n-1} q_k.$$

The aim of the talk is to determine the necessary and sufficient conditions for the weights $\mathbf{q} = \{q_k\}$, ensuring that the sequence of operators $\left\{T_n^{(\mathbf{q})} f\right\}$ associated with Walsh system, is convergent almost everywhere for all integrable function f. We also examines the convergence of a sequence of tensor product operators denoted as $\left\{T_n^{(\mathbf{q})} \otimes T_n^{(\mathbf{p})}\right\}$ involving functions of two variables.

Commutator of nonsingular integral operator on generalized weighted Orlicz-Morrey spaces

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Let $\mathcal{B} = \{B(x,r) : x \in \mathbb{R}^n,\ r > 0\}$, $n \geq 3$. For $1 < p < \infty$, a locally integrable function $w : \mathbb{R}^n \to [0,\infty)$ is said to be an A_p weight if

$$\sup_{B\in\mathcal{B}} \left(\frac{1}{|B|}\int_B w(x)dx\right)\left(\frac{1}{|B|}\int_B w(x)^{-\frac{p'}{p}}dx\right)^{\frac{p}{p'}} < \infty.$$

A locally integrable function $w : \mathbb{R}^n \to [0,\infty)$ is said to be an A_1 weight if

$$\frac{1}{|B|}\int_B w(y)dy \leq Cw(x), \qquad a.e.\ \ x \in B$$

for some constant $C > 0$. We define $A_\infty = \bigcup_{p\geq 1} A_p$.

For any $w \in A_\infty$ and any Lebesgue measurable set E, we write $w(E) = \int_E w(x)dx$.

A function $\Phi : [0,\infty) \to [0,\infty]$ is called a Young function if Φ is convex, left-continuous, $\lim\limits_{r\to+0} \Phi(r) = \Phi(0) = 0$ and $\lim\limits_{r\to\infty} \Phi(r) = \infty$.

We study the boundedness of the commutator of nonsingular integral operator

$$[b,\widetilde{T}]f(x) = \int_{\mathbb{R}^n_+} \frac{(b(x)-b(y))\, f(y)}{|\tilde{x}-y|^n}\, dy \tag{20}$$

on weighted Orlicz spaces $L^{\Phi}_{w}(\mathbb{R}^n_+)$. Here $\tilde{x} = (x', -x_n) \in \mathbb{R}^n_+ = \mathbb{R}^{n-1} \times (0, \infty)$. The weighted Orlicz space $L^{\Phi}_{w}(\mathbb{R}^n_+)$ is equipped with the norm

$$\|f\|_{L^{\Phi}_{w}(\mathbb{R}^n_+)} = \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^n_+} \Phi\Big(\frac{|f(x)|}{\lambda}\Big) w(x) dx \le 1 \right\}.$$

The operator $\widetilde{T}$ and its commutator appear in [1] in connection with boundary estimates for solutions to elliptic equations. In [4] the author studied the boundedness of the nonsingular integral operator $\widetilde{T}$ on weighted Orlicz spaces $L^{\Phi}_{w}(\mathbb{R}^n_+)$.

We recall an important pair of indices used for Young functions. For any Young function Φ, write $h_{\Phi}(t) = \sup_{s>0} \frac{\Phi(st)}{\Phi(s)}, \quad t > 0$. The lower and upper dilation indices of Φ are defined by

$$i_{\Phi} = \lim_{t \to 0^+} \frac{\log h_{\Phi}(t)}{\log t} \text{ and } I_{\Phi} = \lim_{t \to \infty} \frac{\log h_{\Phi}(t)}{\log t},$$

respectively.

Theorem 1. *[5] Let Φ be a Young function, $w \in A_{i_{\Phi}}$ and $[b, \widetilde{T}]$ be a commutator of nonsingular integral operator, defined by (20). If $\Phi \in \Delta_2 \cap \nabla_2$ and $b \in BMO$, then the commutator operator $[b, \widetilde{T}]$ is bounded on $L^{\Phi}_{w}(\mathbb{R}^n_+)$.*

We now define generalized weighted Orlicz-Morrey spaces of the third kind, see for example, [2], [3]. The *generalized weighted Orlicz-Morrey space $M^{\Phi,\varphi}_{w}(\mathbb{R}^n)$ of the third kind* is defined as the set of all measurable functions f for which the norm

$$\|f\|_{M^{\Phi,\varphi}_{w}(\mathbb{R}^n)} \equiv \sup_{x \in \mathbb{R}^n,\, r>0} \varphi(x,r)^{-1}\, \Phi^{-1}\Big(w(B(x,r))^{-1}\Big)\, \|f\|_{L^{\Phi}_{w}(B(x,r))}$$

is finite.

Theorem 2. *Let Φ be a Young function with $\Phi \in \Delta_2 \cap \nabla_2$, $w \in A_{i_{\Phi}}$, $\varphi : \mathbb{R}^n_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ be measurable function such that for all $x \in \mathbb{R}^n$ and $r > 0$*

$$\int_r^{\infty} \Big(1+\ln \frac{t}{r}\Big) \Big(ess \inf_{t<s<\infty} \frac{\varphi_1(x,s)}{\Phi^{-1}\big(w(B(x,s))^{-1}\big)}\Big) \Phi^{-1}\big(w(B(x,t))^{-1}\big) \frac{dt}{t} \le C\varphi(x,r).$$

where $C > 0$ does not depend on x and r. Then for any $f \in M^{\Phi,\varphi}_{w}(\mathbb{R}^n_+)$ and $b \in BMO$ there exist constants depending on n, Φ, φ and the kernel such that

$$\|\widetilde{T} f\|_{M^{\Phi,\varphi}_{w}(\mathbb{R}^n_+)} \le C \|f\|_{M^{\Phi,\varphi}_{w}(\mathbb{R}^n_+)}, \quad \|[b, \widetilde{T}]\|_{M^{\Phi,\varphi}_{w}(\mathbb{R}^n_+)} \le C \|a\|_* \|f\|_{M^{\Phi,\varphi}_{w}(\mathbb{R}^n_+)}.$$

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The Peano differentiability of Riesz potentials

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The quasi-metric on R^n between points $x = (x_1, x_2, ..., x_n)$ and $y = (y_1, y_2, ..., y_n)$ define by

$$\rho(x, y) = \left(\sum_{j=1}^{n} |x_j - y_j|^{\frac{2}{\lambda_j}} \right)^{\frac{1}{2}},$$

where $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n)$ and components of λ are positive numbers. If $\lambda_1 = \lambda_2 = ... = \lambda_n = 1$, then $\rho(x, y)$ turns to the Euclidean metric.

Take donatations $\lambda_{\min} = \min\{\lambda_1, \lambda_2, ..., \lambda_n\}$,$\lambda_{\max} = \max\{\lambda_1, \lambda_2, ..., \lambda_n\}$, $|\lambda| = \lambda_1 + \lambda_2 + ... + \lambda_n$

It is clear that inequality

$$\rho(x, y) \leq 2^{\frac{1}{\lambda_{\min}}} (\rho(x, z) + \rho(y, z))$$

holds for any $x, y, z \in R^n$.

Define ball $B(x, r) = \{y \in R^n : \rho(x, y) < r\}$ centered at$x = (x_1, x_2, ..., x_n)$ and with radius $r > 0$.

Let $0 < \alpha < n$. Operator

$$I_\alpha f(x) = \int_{R^n} \rho(x, y)^{\alpha - n} f(y)\, dy \tag{1}$$

is called α-potential or Riesz potential. If$\rho(x, y)$ is the Euclidean metric, then we have the classical Riesz potential.

If f is a local integrable and nonnegative function, then it is known that integral (1) is finite a.e. if and on if

$$\int_{\mathrm{R}^n} (1 + \rho(0, y))^{\alpha - n} f(y) dy < \infty$$

Let $p = \frac{n}{\alpha} > 1$ and $w(t)$ be a positive, increasing function in $(0, \infty)$ with condition

$$\int_1^\infty w^{-\frac{1}{p-1}}(t)\, t^{-1} dt < \infty$$

and there exists number $D > 0$ such that for any $t > 0$the inequality

$$w(2t) < Dw(t)$$

holds.

As examples of w satisfying the above conditions can be taken

$$w(t) = \ln^\nu(2 + t);$$

$$w(t) = \ln^{p-1}(2 + t) \ln^\nu(2 + \ln(2 + t));$$

and etc., where $\nu > p - 1 > 0$.

Determine $\Phi(t) = t^p w\,(t)$. By $\Psi_{p,w}$ we denote a class of all local integrable and nonnegative $f : \mathrm{R}^n \to [0, \infty)$ satisfying property

$$\int_{\mathrm{R}^n} \Phi\,(f\,(y))\,dy < \infty.$$

Let function$f$ be defined on the neighborhood of point $x_0 \in R^n$ and there exists polynomial $P(x)$ of degree less than or equal m such that

$$\lim_{\rho(x,y)\to 0} \rho(x, y)^{-m}\,[f(x) - P(x)] = 0.$$

Then we call f is the m times Peano differentiable at point x_0.

Let $1 < p < \infty$ and $0 < \alpha < n$ For set $X \in \mathrm{R}^n$ define

$$C_{\alpha,p}\,(X) = \inf_f \|f\|_p^p\,,$$

where infimum is taken over all nonnegative functions $f \in L^p\,(R^n)$ such that $I_\alpha f\,(x) \geq 1$ for any $x \in X$. Quantity$C_{\alpha,p}\,(E)$ is called a (α, p)-capacity or Riesz capacity of$X$. If a property holds on R^n except for a set (α, p)-capacity zero, then we say that the property holds $C_{\alpha,p}$-q.e.

Theorem. *Let be given function$f \in \Psi_{p,w}$. Suppose that $I_\alpha f(x)$ is finite a.e., m is a natural number, $m \leq \alpha$, $\frac{2}{\lambda_i}$ is a natural number or $\frac{2}{\lambda_i} > m$, for $i = \overline{1, n}$, and*

$$\lambda_{\max}\frac{m}{\alpha} < 1 < \lambda_{\min}\frac{m+1}{m}.$$

Then $I_\alpha f(x)$ is $C_{\alpha-\,\lambda_{\max} m,p}$-q.e. mtimes Peano's differentiable.

Characterization of G-Lipschitz spaces via commutators of the G-maximal function

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Let M_G be the Gegenbauer maximal (G-maximal) function an b be a locally integrable function. Denote M_G^b the G-maximal commutator of M_G with b. In this paper we consider the boundedness M_G^b on Lebesgue and G-Morrey spaces when b belong to the Gegenbauer-Lipschitz (G-Lipschitz) space, by which some characterization of the Gegenbauer –Lipschitz spaces are given.

Let $L_p(\mathbb{R}_+, G) = L_{p,\lambda}(\mathbb{R}_+)$, $\mathbb{R}_+ = [0, \infty)$ be the space of $\mu(x) = sh^{2\lambda}x$ measurable function on $\mathbb{R}_+$ with the finite norm

$$\|f\|_{L_{p,\lambda}(\mathbb{R}_+)} = \left(\int_{\mathbb{R}_+} |f(chx)|^p \, d\mu_\lambda(x)\right)^{\frac{1}{p}}, \quad 1 \le p < \infty,$$

$$\|f\|_{L_{\infty,\lambda}(\mathbb{R}_+)} = \|f\|_\infty = ess\sup_{x\in\mathbb{R}_+} |f(chx)|, \; p = \infty.$$

For $f \in L_{1,\lambda}^{loc}(\mathbb{R}_+)$ the G-fractional maximal operator $\mathfrak{M}_G^\alpha$ is defined as follows:

$$\mathfrak{M}_G^\alpha f(chx) = \sup_{r>0} |H_r|_\lambda^{\frac{\alpha}{\gamma}-1} \int\limits_{H_r} A_{chy}^\lambda |f(chx)| \, d\mu_\lambda(y), \quad 0 < \alpha < \gamma.$$

$\mathfrak{M}_G^0 \equiv M_G$, and $|H_r|_\lambda = \int\limits_0^r d\mu(x)$.

$$J_G^\alpha f(chx) = \int\limits_0^\infty A_{chy}^\lambda f(chx) \Big(sh\frac{y}{2}\Big)^{\alpha-\gamma} d\mu_\lambda(y), \quad 0 < \alpha < \gamma$$

corresponding.

Let $b \in L_{1,\lambda}^{loc}(\mathbb{R}_+)$, then commutator generated by function b and the $\mathfrak{M}_G^{\alpha}$ and also J_G^{α} are defined as follows:

$$\mathfrak{M}_G^{b,\alpha} f(chx) = \sup_{r>0} |H_r|_{\lambda}^{\frac{\alpha}{\gamma}-1} \int\limits_{H_r} \left| A_{chy}^{\lambda} b(chx) - b_{H_r}(chx) \right| A_{chy}^{\lambda} \left| f(chx) \right| d\mu_{\lambda}(y),$$

moreover $\mathfrak{M}_G^{b,0} \equiv M_G^b$.

By analogy with the classical case we introduce the G-Lipschitz space as follows.

Definition 1. *Let $0 < \beta \leq 1$, we say a function f belong to the G-Lipschitz space $\dot{\Lambda}_{\beta}(\mathbb{R}_+, G)$ if there exists a constant C such that for all $x, y \in \mathbb{R}_+$*

$$\dot{\Lambda}_{\beta}(\mathbb{R}_+, G) = \left\{ f : |A_{chy}^{\lambda} f(chx) - f(chx)| < C(chy-1)^{\beta} \right\},$$

and the smallest such constant C we called the $\dot{\Lambda}_{\beta}(\mathbb{R}_+, G)$ norm of f and denote $\|f\|_{\dot{\Lambda}_{\beta}(\mathbb{R}_+, G)}$.

The following theorems are analogues of Theorems 1, 2, and 3 in [1].

Our first result can be stated as follows.

Theorem 2. Let b be a locally integrable function and $0 < \beta < 1$. Then the following statements are equivalent:

(1) $b \in \dot{\Lambda}_{\beta}(\mathbb{R}_+, G)$,

(2) M_G^b is bounded from $L_{p,\lambda}(\mathbb{R}_+, G)$ to $L_{q,\lambda}(\mathbb{R}_+)$ for $1 < p < \gamma/\beta$ and $1/p - 1/q = \beta/\gamma$,

(3) M_G^b satisfies the weak-type $(1, \gamma/\gamma - \beta)$ estimates, namely there exists a positive constant C such that for all $\nu > 0$

$$\left| \left\{ x \in \mathbb{R}_+ : M_G^b f(chx) > \nu \right\} \right|_{\lambda} \leq C \left(\nu^{-1} \|f\|_{L_{1,\lambda}(\mathbb{R}_+)} \right)^{\frac{\gamma}{\gamma-\beta}}.$$

Definition 3. *Let $1 \leq p < \infty$ and $0 \leq \nu \leq \gamma$. We denote by $L_{p,\lambda,\nu}(\mathbb{R}_+)$ the Gegenbauer-Morrey (G-Morrey) space associated with the Gegenbauer differential operator G as the set of locally integrable functions $f(chx)$, $x \in \mathbb{R}_+$ with the finite norm*

$$\|f\|_{L_{p,\lambda,\nu}(\mathbb{R}_+)} = \sup_{\substack{x \in \mathbb{R}_+ \\ r>0}} \left(|H_r|_{\lambda}^{-\nu/\gamma} \int\limits_0^r A_{chy}^{\lambda} |f(chx)|^p \, d\mu_{\lambda}|(y) \right)^{\frac{1}{p}}.$$

Theorem 4. *(Adams type result) Let b be a locally integrable function and $0 < \beta < 1$. Suppose that $1 < \beta < \gamma/\beta$, $0 < \nu < \gamma - \beta$ and $1/p - 1/q = \beta/(\gamma - \nu)$. Then $b \in \dot{\Lambda}_\beta(\mathbb{R}_+, G)$ if and only if M_G^b is bounded from $L_{p,\lambda,\nu}(\mathbb{R}_+)$ to $L_{q,\lambda,\nu}(\mathbb{R}_+)$.*

Theorem 5. *(Spanne type result) Let b be a locally integrable function and $0 < \beta < 1$. Suppose that $1 < p < \gamma/\beta$, $0 < \nu < \gamma - \beta p$, $1/p - 1/q = \beta/\gamma$ and $\nu/p = \mu/q$. Then $b \in \dot{\Lambda}_\beta(\mathbb{R}_+, G)$ if and only if M_G^b is bounded from $L_{p,\lambda,\nu}(\mathbb{R}_+)$ to $L_{q,\lambda,\mu}(\mathbb{R}_+)$.*

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On basicity of some trigonometric system in banach function spaces

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In this work, it is considered the trigonometric system $\{1; \cos nx;\ x \sin nx\}_{n\in\mathbb{N}}$, which is a collection of eigenfunctions of one nonlocal spectral problem for an ordinary second-order differential operator. Let $X(-\pi,\pi)$ be a Banach Function Space (by Luxembourg classification) on $(-\pi,\pi)$ with Lebesgue measure. A criterion is obtained for the trigonometric system $\left\{\frac{1}{2}; \cos nt; \sin nt\right\}_{n\in\mathbb{N}}$ to have the Riesz Property in $X(-\pi,\pi)$. It is proved that if the trigonometric system has the Riesz Property in $X(-\pi,\pi)$, then the system (T) also forms a basis for $X(-\pi,\pi)$.

Assume some notations. $\mathbb{N}$ be the set of natural numbers, $\mathbb{Z}_+ = \{0\} \cup \mathbb{N}$; $J = (0, 2\pi)$. $\mathbb{R}$ be the set of real numbers; $\mathbb{C}$ be the set of complex numbers; ($K \equiv \mathbb{R}$ or $K \equiv \mathbb{C}$). $C_0^\infty(J)$ is the set of all infinitely differentiable functions on J with compact support in J.

Consider the following trigonometric system

$$(T) = \left\{u_n^+(x); u_{n+1}^-(x)\right\}_{n \in \mathbb{Z}_+},$$

where

$$u_n^+(x) \equiv \cos nx;\ u_{n+1}^-(x) = x \sin(n+1)x,\ n \in \mathbb{Z}_+.$$

Let us recall the definition of BFS.

Definition 1. [1] $\|\cdot\|_{X(J)} : L_0(J) \to \bar{R}_+ = [0, +\infty]$ *is called a Banach Function norm, iff:*

i) $\|\cdot\|_{X(J)}$ *is a norm on* $L_0(J)$*;*

ii) $f; g \in L_0(J) : |f| \leq |g|$ *a.e. on* $J \Rightarrow \|f\|_{X(J)} \leq \|g\|_{X(J)}$*;*

iii) Fatou property. $|f_n| \uparrow |f|$, $n \to \infty \Rightarrow \|f_n\|_{X(J)} \uparrow \|f\|_{X(J)}$*;*

iv) $\forall E \subset J$ *(measurable in Lebesgue sense)*$\Rightarrow \|\chi_E\|_{X(J)} < +\infty$*;*

v) it holds the continuous embedding $X(J) \subset L_1(J)$*,*

where $L_p(J)$, $1 \leq p < +\infty$, *denotes the ordinary Lebesgue space with the norm*

$$\|f\|_{L_p(J)} = \left(\int_J |f|^p\, dt\right)^{\frac{1}{p}}.$$

BFS $X(J)$ is defined by

$$X(J) = \left\{f \in L_0(J) : \|f\|_{X(J)} < +\infty\right\}.$$

$X(J)$ with the norm $\|\cdot\|_{X(J)}$ is a Banach space.

Assume the following

Definition 2. [1] *We will call BFS* $X(J)$ *a* (R)*-space, if* $f(\cdot) \in X(J) \Rightarrow f(2\pi - \cdot) \in X(J)$.

Regarding the classical trigonometric system

$$\left\{\frac{1}{2}; \cos nx; \sin nx\right\}_{n \in N}. \tag{1}$$

in BFS $X(-\pi, \pi)$ it is true so-called the Riesz Property, which we define as follows:

Definition 3. *We will say that the trigonometric basis (1) of BFS* $X(-\pi,\pi)$ *has the Riesz property in* $X(-\pi,\pi)$ *if* $\exists C>0$: *for* $\forall f \in X(-\pi,\pi)$ *it holds*

$$\left.\begin{array}{l} \|S_n^c(f)\|_{X(-\pi,\pi)} \le C\|f\|, \forall n \in \mathbb{Z}_+, \\ \|S_n^s(f)\|_{X(-\pi,\pi)} \le C\|f\|, \forall n \in \mathbb{N}. \end{array}\right\} \tag{2}$$

We've introduced one method to establish the basicity of the system (T) in BFS $X(J)$. The following theorem is true.

Theorem 1. *Let BFS* $X(J)$ *be a* (R)*-space, in which the set* $C_0^\infty(J)$ *is dense and the trigonometric system (1) form a basis for it. Then the system* (T) *also forms a basis for* $X(J)$ *and it has the Riesz Property, i.e.* $\exists C>0$, $\forall f \in X(J)$ *it holds*

$$\left\|\sum_{k=0}^{n} e_k^+(f)\cos kx\right\|_{X(J)} \le C\|f\|_{X(J)}, \ \forall n \in \mathbb{Z}_+,$$

$$\left\|\sum_{k=1}^{n} e_k^-(f)\ x\ \sin kx\right\|_{X(J)} \le C\|f\|_{X(J)}, \ \forall n \in \mathbb{N},$$

where $\left\{e_0^+; e_k^+; e_k^-\right\}_{k\in\mathbb{N}} \subset X^*(J)$ *is a system biorthogonal to the basis* $(\mathcal{T})$.

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On the boundedness of a singular operator in grand spaces $L_{p),\nu)}(-\pi,\pi)$

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Let $l_{p)}$, $1<p<+\infty$, be the grand space of sequences of scalars $\{a_n\}_{n\in Z}$, such that

$$\left\|\{a_n\}_{n\in Z}\right\|_{p)} = \sup_{0<\varepsilon<p-1} \varepsilon^{\frac{1}{p(\varepsilon)}} \left(\sum_{n=-\infty}^{+\infty} |a_n|^{p'(\varepsilon)}\right)^{\frac{1}{p'(\varepsilon)}} < +\infty, \tag{21}$$

where $p(\varepsilon)=p-\varepsilon$, $p'(\varepsilon)=\frac{p(\varepsilon)}{p(\varepsilon)-1}$. The space $l_{p)}$ is a Banach space with the usual linear operations and norm (21). Let us denote by $g_{p)}$ the closure in the space $l_{p)}$ of the set c_{00} of finite sequences of numbers.

The following theorem gives a characterization of the space $g_{p)}$.

Theorem 1. *Let $a=\{a_n\}_{n\in Z}\in l_{p)}$, $1<p<+\infty$. Then the following conditions are equivalent:*

i) $a\in g_{p)}$;

ii) *the equality* $\lim\limits_{\varepsilon\to+0} \varepsilon^{\frac{1}{p(\varepsilon)}} \left(\sum\limits_{n=-\infty}^{+\infty} |a_n|^{p'(\varepsilon)}\right)^{\frac{1}{p'(\varepsilon)}} = 0$ *is true;*

iii) *the equality* $\lim\limits_{m\to+\infty} \sup\limits_{0<\varepsilon<p-1} \varepsilon^{\frac{1}{p(\varepsilon)}} \left(\sum\limits_{|n|\ge m+1}^{+\infty} |a_n|^{p'(\varepsilon)}\right)^{\frac{1}{p'(\varepsilon)}} = 0$ *is true.*

Let $L_{p)}(-\pi,\pi)$ be a grand Lebesgue space ([1]). Similar to space $L_{p,\nu}(-\pi,\pi)$ ([2]), by $L_{p),\nu)}(-\pi,\pi)$ we denote the space of functions $f\in L_{p)}(-\pi,\pi)$ such that $\hat{f}=\{c_n\}_{n\in Z}\in l_{\nu)}$, where $c_n=\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int}dt$, $n\in Z$. The space $L_{p),\nu)}(-\pi,\pi)$ is a Banach space with norm $\|f\|_{p),\nu)}=\|f\|_{p)}+\left\|\hat{f}\right\|_{\nu)}$. Let $G_{p)}(-\pi,\pi)$ be the closure in $L_{p)}(-\pi,\pi)$ of the set of infinitely differentiable functions $C_0^\infty([-\pi,\pi])$ ([3]). Consider the subspace $G_{p),\nu)}(-\pi,\pi)$ of the space $L_{p),\nu)}(-\pi,\pi)$ of functions $f\in G_{p)}(-\pi,\pi)$ for which $\hat{f}\in g_{\nu)}$. In [2] it is shown that the system $\{e^{int}\}_{n\in Z}$ forms a basis in $L_{p,\nu}(-\pi,\pi)$. In this paper, using the basis property of the system $\{e^{int}\}_{n\in Z}$ in $G_{p)}(-\pi,\pi)$ ([4, 5]), this issue is studied in the space $G_{p),\nu)}(-\pi,\pi)$.

The following theorem holds.

Theorem 2. *The system* $\{e^{int}\}_{n\in Z}$ *forms a basis in* $G_{p),\nu)}(-\pi,\pi)$, $1<p,\nu<+\infty$*, and in* $L_{1,\nu)}(-\pi,\pi)=\left\{f\in L_1(-\pi,\pi): \hat{f}\in l_{\nu)}\right\}$, $2<\nu<+\infty$.

Let $\gamma=\{z:|z|=1\}$ be the unit circle and $T:[-\pi,\pi]\to\gamma$ be the identification operator: $T(t)=e^{it}$. Let us denote by $G_{p),\nu)}(\gamma)$ the space of functions f given on γ, such that $f(Tt)\in G_{p),\nu)}(-\pi,\pi)$.

The following theorem establishes the boundedness in the space $G_{p),\nu)}(\gamma)$ of a singular operator with a Cauchy kernel.

Theorem 3. *Singular operator* S_γ *defined by the formula*

$$S_\gamma(f)(z)=\frac{1}{2\pi i}\int_\gamma\frac{f(\xi)}{\xi-z}d\xi, z\in\gamma,$$

is bounded in $G_{p),\nu)}(\gamma)$, $1<p,\nu<+\infty$.

Note that the boundedness of the operator S_γ in the space $L_{p,\nu}(-\pi,\pi)$ was studied in [6].

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Some aspects of regular and rapid variation

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We review some known and some new results on translationally regularly varying and translationally rapidly varying functions and sequences, important objects in the asymptotic analysis of divergent processes. Among others, we provide representation theorems, Galambos-Bojanić-Seneta type results, selection principles and fixed point results related to these classes of functions and sequences.

Some remark on laplace equation in grand Lebesgue-Hardy classes

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In this report, we announce some results on solvability and Noetherness for place-Laplace equation in grand Lebesgue-Hardy classes defined on the unit disk.

We use the following notations: C - the set of all complex numbers, R - the set of all real numbers, $D = \{z \in C : |z| < 1\}$ - the unit disk, $T = \{z : |z| = 1\}$ - the unit circle. We will identify arbitrary function $f : T \to C$ with the function defined on $[-\pi;\pi)$ as follows: $f : [-\pi;\pi) \to C$ (or R) $\Leftrightarrow f(t) := f(e^{it})$, and we will assume that it is extended periodically to the whole of R: $f(t+2\pi) = f(t)$. For $f : D \to R$, we will consider the following family of functions: $f_r(t) = f(re^{it})$, $0 \le r < 1$, $t \in [-\pi;\pi)$, where (r,t) - is a polar coordinates. $H(D) = \{u : D \to R : \Delta u = 0 \text{ in } D\}$ - denotes the class of all harmonic functions in the unit disk.

Grand Lebesgue space $L_{p)}(-\pi;\pi)$, $1 < p < +\infty$ is a space of all measurable functions with the norm $\|f\|_{p)} = \sup\limits_{0<\varepsilon<p-1} \left\{ \left(\varepsilon \int_{-\pi}^{\pi} |f|^{p-\varepsilon} dt \right)^{\frac{1}{p-\varepsilon}} \right\} < +\infty$. It is well known that $L_{p)}(\Omega)$ is a nonseparable rearrangement-invariant Banach function space.

Shift operator $(T_\delta f)(.) = f(.+\delta)$ is defined as

$$(T_\delta f)(x) = \begin{cases} f(x+\delta), & x+\delta \in \Omega, \\ 0, & x+\delta \notin \Omega. \end{cases}$$

The subspace $N_{p)}(-\pi;\pi)$ is a subspace of all functions from $L_{p)}(\Omega)$ where the shift operator acts continuously, i.e.

$$N_{p)}(-\pi,\pi) = G_{p)}(-\pi,\pi)) = \left\{ f \in L_{p)} : \|f(.+\delta) - f(.)\|_{p)} \xrightarrow[\delta\to 0]{} 0 \right\} =$$

$$= \left(L_{p)}(T)\right)_a = \left(L_{p)}(T)\right)_b = \overline{C_0^\infty(-\pi;\pi)},$$

where $\left(L_{p)}(T)\right)_a$; $\left(L_{p)}(T)\right)_b$ are the closures of all absolutely continuous and bounded functions in a topology of $L_{p)}(T)$, respectively.

The class $H(D)$ is a class of all harmonic functions on D: $H(D) = \{u : D \to R : \Delta u = 0 \text{ in } D\}$.

Define **the spaces** $h_p(D)$, $h_{p)}(D)$ as follows

$$h_p(D) = \left\{ u \in H(D) : \sup_{0\le r<1} \|u_r(.)\|_{L_p(T)} < +\infty \right\},$$

$$h_{p)}(D) = \left\{ u \in H(D) : \sup_{0\le r<1} \|u_r(.)\|_{L_{p)}(T)} < +\infty \right\},$$

with the norms

$$\|u\|_{h_p} = \sup_{0\le r<1} \|u_r(.)\|_{L_p(T)}, \; \|u\|_{h_{p)}} = \sup_{0\le r<1} \|u_r(.)\|_{L_{p)}(T)},$$

Hardy classes $h_p^1(D)$, $h_{p)}^1(D)$ are

$$h_p^{(1)}(D) = \left\{ u \in h_p : \frac{\partial u}{\partial r}, \frac{\partial u}{\partial \varphi} \in h_p \right\},\ h_{p)}^{(1)}(D) = \left\{ u \in h_{p)} : \frac{\partial u}{\partial r}, \frac{\partial u}{\partial \varphi} \in h_{p)} \right\},$$

with the norms

$$\|u\|_{h_p^{(1)}(D)} = \|u\|_p + \left\| \frac{\partial u}{\partial r} \right\|_p + \left\| \frac{\partial u}{\partial \varphi} \right\|_p,\ \|u\|_{h_{p)}^{(1)}(D)} = \|u\|_{p)} + \left\| \frac{\partial u}{\partial r} \right\|_{p)} + \left\| \frac{\partial u}{\partial \varphi} \right\|_{p)},$$

respectively.

Let $A(D)$ be the class of all analytic functions on D. We define **Hardy classes** $H_p^+(D)$, $H_{p)}^+(D)$ as follows

$$H_p^+(D) = \left\{ f \in A(D) : \|f\|_{H_p^+} = \sup_{0<r<1} \|f_r(.)\|_{L_p} < +\infty \right\}, H_{p)}^+(D) =$$

$$= \left\{ f \in A(D) : \|f\|_{H_{p)}^+} = \sup_{0<r<1} \|f_r(.)\|_{L_{p)}} < +\infty \right\}.$$

It is clear that $H_p^+(D) = \{ f \in A(D) : Ref, Imf \in h_p \}$.

Denote by γ the following **trace operator** :$\gamma : h_{p)}^+(D) \to L_{p)}(T) : \gamma(u) = u^+$, where u^+ is a nontangential boundary function.

Let us consider the subspaces

$$Nh_{p)}(D) = \left\{ f \in h_{p)}(D) : f^+ \in N_{p)}(T) \right\}, NH_{p)}^+(D) =$$

$$= \left\{ f \in H_{p)}^+(D) : f^+ \in N_{p)}(T) \right\}.$$

The following two problem are considered

Problem I.

$$\left. \begin{array}{l} \Delta u = 0,\ in\ D, \\ \gamma^+ u = f,\ on\, T \end{array} \right\},$$

where $u \in h_{p)}(D)$, $f \in X(T)$.

Problem II. Let (r, φ), $0 \le r < 1$, $-\pi \le \varphi < \pi$, in D. Consider the place-Laplace operator in polar coordinates

$$\Delta_{r,\varphi} u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2},$$

and the following boundary value problem with oblique derivatives

$$\Delta_{r,\varphi} u = 0, \;\; u \in Nh^{(1)}_{p)}(D), \;\; \gamma\left(\cos\varphi\frac{\partial u}{\partial r} + \sin\varphi\frac{\partial u}{\partial \varphi}\right) \equiv$$

$$\equiv \left(\cos\varphi\frac{\partial u}{\partial r} + \sin\varphi\frac{\partial u}{\partial \varphi}\right)|_{r=1} = f(\varphi) \in N_{p)}(T), \;\; \varphi \in [-\pi; \pi).$$

Theorem 1. *Let* $1 < p < +\infty$. *Then for* $\forall f \in L_{p)}(T)$ *the Dirichlet Problem* I *has a unique solution in* $h_{p)}(D)$. *Moreover, the relation* $\|u\|_{h_{p)}} = \|f\|_{L_{p)}}$, *holds.*

Theorem 2. *Problem II is a Noetherian and its index*χ *is* $\chi = -2$.

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The estimates of approximation by weighted modulus of continuity in Lebesgue spaces

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In this abstract, we consider estimates of approximation of functions with respect to a suitable weighted norm via the new type model of continuity. First, we give that the weighted modulus of continuity has properties similar to the usual modulus of continuity defined by

$$\Omega_p(f;\delta) = \sup_{0<h\le\delta}\left(\int_0^\infty \left(\frac{|f(x+h)-f(x)|}{(1+x^2)(1+h^2)}\right)^p dx\right)^{\frac{1}{p}}.$$

We denote the set of functions that satisfy this inequality by B_ω to obtain:

$$B_\omega(R_+) := \{f : |f(x)| \le M_f \omega(x)\}$$

Then we define

$$C_\omega(R_+) := \{f : f \in B_\omega \text{ and f is continuous }\}$$

$$C_{\omega}^{K}\left(R_{+}\right) := \left\{ f : f \in C_{\omega}\left(R_{+}\right) \text{ and } \lim_{x\to+\infty} \frac{f(x)}{\omega(x)} = K_f < \infty \right\}$$

and

$$C_{\omega}^{0}\left(R_{+}^{2}\right) := \left\{ f : f \in C_{\omega}^{K}\left(R_{+}^{2}\right) \text{ and } \lim_{x\to+\infty} \frac{f(x)}{\rho(x)} = 0 \right\}.$$

It is well known that the norm in B_{ω} is defined as

$$\|f\|_{\omega} = \sup_{x\in R_{+}} \frac{|f(x)|}{\omega(x)}.$$

We refer reader to the works [1] and [2].

Let $x \in (0, \infty)$ and let $\omega(x) = 1 + x^2$.

Now we give the main result of this abstract.

Theorem 1. *Let $\omega(x) = 1+x^2$ is weighted funciotions and B_n be a sequence of positive linear operators from $L_{p,\omega}\left(R_{+}\right)$ to $B_{\omega}\left(R_{+}\right)$. If*

$$\left\|B_n\left(t^m; x\right) - x^m\right\|_{L_{p,\omega}} \to 0, \quad \text{as } n \to \infty,\ m = 0, 1, 2$$

then for all functions $f \in L_{p,\omega}\left(R_{+}\right)$

$$\left\|B_n f - f\right\|_{L_{p,\omega}} \to 0, \text{ as } n \to \infty.$$

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Weighted inequalities for Fourier series with respect to the multiplicative system with coefficients of bounded variation

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In this work, we consider the problem of weighted inequalities for Fourier series with respect to the the multiplicative systems with coefficients of bounded variation.

Let $\mathbb{Z}$ be the set of integers. Let $\{p_k\}_{k=1}^{\infty}$ is a sequence of natural numbers $p_k \geq 2$, $k \in \mathbb{N}$, $\sup\limits_k p_k = N < \infty$. and $m_0 = 1$, $m_n = p_1 p_2 \cdots p_n$, $n \in \mathbb{N}$. For $x \in [0,1)$ with the decomposition $x = \sum_{k=1}^{\infty} \frac{x_k}{m_k}$, $x_k \in \mathbb{Z} \cap [0, p_k)$, and for $n \in \mathbb{Z}_+$ with the representation $n = \sum_{j=1}^{\infty} \alpha_j m_{j-1}$, $\alpha_j \in \mathbb{Z} \cap [0, p_j)$ the multiplicative system $\{\psi_n\}_{n=0}^{\infty}$ [1] is define by

$$\psi_n(x) = exp\left(2\pi i \sum_{j=1}^{\infty} \frac{\alpha_j x_j}{p_j}\right), \quad n \in \mathbb{Z}_+.$$

Let $1 \leq p < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$, and $\varphi(x) \geq 0$ be some locally integrable function on $[0,1]$. A Lebesgue measurable function $f(x)$ belongs to the space $L_{p,\varphi}$ if

$$\|f\|_{p,\varphi} = \left(\int_0^1 |f(x)\,\varphi(x)|^p\,dx\right)^{\frac{1}{p}} < \infty.$$

We define, that $w \in RB_p$, if for any $x \in (0,1)$

$$\frac{1}{x^p}\int_0^x \omega(t)\,dt \leq \int_x^1 \frac{\omega(t)}{t^p}\,dt.$$

Let u_n, v_n be nonnegative sequences. We put $f(x) = \sum_{k=0}^{\infty} \lambda_k \psi_k(x)$.

Theorem 1. *Let* $0 < p < \infty$ *and* $\Lambda_n = \sum_{k=n}^{\infty} |\lambda_k - \lambda_{k+1}|$. *Then the inequality*

$$\|f\|_{L^p(\omega(0,1))} \leq \left(\sum_{n=1}^{\infty} \Lambda_n^p \int_{\frac{1}{n+1}}^{\frac{1}{n}} \frac{\omega(x)}{x^p}\,dx\right)^{1/p}$$

holds if and only if $\omega \in RB_p$.

Theorem 2. Let $0 < p \le q < \infty$ and $\Lambda_n = \sum_{k=n}^{\infty} |\lambda_k - \lambda_{k+1}|$. Let also

$$I_1 := \sup_{n\in\mathbb{N}} \left(\sum_{\nu=1}^{n} u_\nu \right)^{-1/p} \left(\int_{\frac{1}{n}}^{1} \frac{\omega(x)}{x^q}\, dx \right)^{1/q},$$

$$I_2 := \sup_{n\in\mathbb{N}} \left(V_n \left(\int_0^{1/n} \omega(x)\, dx \right)^{1/q} \right),$$

where

$$V_n = \begin{cases} \left(\sum_{\nu=1}^{n-1} \nu^{p'} \alpha_\nu + \frac{n^{p'}}{U_n^{p'/p}} \right)^{1/p}, & p > 1, \\ \frac{n}{U_n^{1/p}}, & p \le 1 \end{cases}$$

$$\alpha_\nu = \frac{1}{U_\nu^{1/p}} - \frac{1}{U_{\nu+1}^{1/p}}, U_\nu = \sum_{k=1}^{\nu} u_k, \quad u_k \ge 0,\ u_1 > 0.$$

If $I_1 + I_2 < \infty$, then

$$\|f\|_{L^q(\omega(0,1))} \le \left(\sum_{n=1}^{\infty} u_n \Lambda_n^p \right)^{1/p}.$$

For the case of trigonometric series such problems were considered in [2].

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Interpolation theorems in Besov-Morrey spaces with dominant mixed derivatives

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In the abstract, we study with the help of the method of integral representation of differential and differential-difference properties functions from intersections of Besov-Morrey spaces with dominant mixed derivatives

$$S^{l^\mu}_{p_\mu,\theta_\mu,\varphi,\beta}B(G_\varphi)(\mu=1,2,\ldots,N) \tag{1}$$

was introduced in paper [1]. Note that in paper [1] was studied embedding theorems in $S^{l}_{p,\theta,\varphi,\beta}B(G_\varphi)$.

Let $G\subset\mathbb{R}^n$, $1\le p_\mu<\infty$, $1\le\theta_\mu\le\infty$ $(\mu=1,2,...,N)$; $\varphi(t)=(\varphi_1(t_1),\varphi_2(t_2),\ldots,\varphi_n(t_n))$, $\varphi_j(t_j)>0$ $(t_j>0)$, $j\in e_n=\{1,2,...,n\}$ are continuously differentiable functions and $\varphi'_j(t_j)>0$ $(t_j>0, j\in e_n)$ and $\lim\limits_{t_j\to+0}\varphi_j(t_j)=0$, $\lim\limits_{t_j\to+\infty}\varphi_j(t_j)=L_j\le\infty$, $(j\in e_n)$. We denote the set of such vector- functions φ by A.

Definition 1. [1] Denote by $S^{l}_{p,\theta,\varphi,\beta}B(G_\varphi)$ the Besov-Morrey spaces with dominant mixed derivatives the Banach space of locally summable functions on G with a finite norm

$$\|f\|_{S^{l}_{p,\theta,\varphi,\beta}B(G_\varphi)}=\sum_{e\subseteq e_n}\left\{\int\limits_{0^e}^{t_0^e}\left[\frac{\|\triangle^{m^e}(\varphi(t),G_{\varphi(t)})D^{k^e}f\|_{p,\varphi,\beta}}{\prod\limits_{j\in e}(\varphi_j(t))^{l_j-k_j}}\right]^\theta\prod_{j\in e}\frac{d\varphi_j(t)}{\varphi_j(t)}\right\}^{\frac{1}{\theta}},$$

where

$$\|f\|_{p,\varphi,\beta;G}=\|f\|_{L_{p,\varphi,\beta}(G)}=\sup_{\substack{x\in G,\\ t_j>0,\\ j\in e_n}}\left(|\varphi([t]_1)|^{-\beta_j}\|f\|_{L_p(G_{\varphi(t)}(x))}\right),$$

and let $m_j > 0, k_j \geq 0$ are integers, $l_j > 0$, and $m_j > l_j - k_j > 0$, $(j \in e_n)$; $e \subseteq e_n, m^e = (m_1^e, m_2^e, \ldots, m_n^e)$, $m_j^e = m_j (j \in e)$, $m_j^e = 0$ $(j \in e_n - e = e')$ $|\varphi([t]_1)|^{-\beta} = \prod\limits_{j \in e_n} \varphi_j([t_j]_1)^{-\beta_j}$, $\beta_j \in [0,1]$, $[t_j]_1 = \min\{1, t_j\}$, $1 \leq \theta \leq \infty$, and let for all $x \in \mathbb{R}^n$

$$G_{\varphi(t)}\left(x\right) = G \cap \left\{ y : |y_j - x_j| < \frac{1}{2}\varphi_j(t_j), j \in e_n \right\},$$

$$\triangle^{m^e}(\varphi(t)) f(x) = \left(\prod_{j \in e} \triangle_j^{m_j}(\varphi_j(t_j)) \right) f(x),$$

and $t_0 = (t_{01}, \ldots, t_{0n})$ is a fixed positive vector, and

$$\int\limits_{a^e}^{b^e} f(x) dx^e = \left(\prod_{j \in e} \int\limits_{a_j}^{b_j} dx_j \right) f(x),$$

i.e., integration is carried out only with respect to the variables x_j whose indices belong to e.

In this work, the following results are studied.

1. Embedding theorem of type

$$D^\nu : \bigcap_{\mu=1}^{N} S^{l_\mu}_{p_\mu, \theta_\mu, \varphi, \beta} B(G_\varphi) \to L_{q,\psi,\beta_1}(G), \psi \in A,$$

2. For generalized mixed derivatives of functions $D^\nu f(x)$ from the intersections of these spaces, generalized Hölder inequality is proved.

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Approximation of functions by singular integrals in the terms of higher order mean oscillation

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Let R^n be n-dimensional Euclidean space of points $x = (x_1, x_2, ..., x_n)$, $B(a,r) := \{x \in R^n\colon |x-a| \le r\}$ –be a closed ball in R^n of radius $r > 0$ centered at the point $a \in R^n$, N be the set of all natural numbers, $\nu = (\nu_1, \nu_2, ..., \nu_n)$, $x^\nu = x_1^{\nu_1} \cdot x_2^{\nu_2} \cdots x_n^{\nu_n}$, $|\nu| = \nu_1 + \nu_2 + ... + \nu_n$, where $\nu_1, \nu_2, ..., \nu_n$ are non-negative integers. Denote the class of all locally p-power summable functions defined on R^n by $L^p_{loc}(R^n)$ $(1 \le p < \infty)$, the class of all locally bounded functions defined on R^n by $L^\infty_{loc}(R^n)$.

Let $f \in L^1_{loc}(R^n)$, $k \in \mathrm{N} \bigcup \{0\}$. Let us consider the polynomial [1]

$$P_{k,\mathrm{B}(a.r)} f(x) := \sum_{|\nu| \le k} \left(\frac{1}{|\mathrm{B}(a,r)|} \int_{\mathrm{B}(a,r)} f(t)\, \varphi_\nu \left(\frac{t-a}{r} \right) dt \right) \varphi_\nu \left(\frac{x-a}{r} \right),$$

where $|B(a,r)|$ denotes the volume of the ball $B(a,r)$ and $\{\varphi_\nu\}$, $|\nu| \le k$, is an orthonormed system obtained from applications of orthogonalization process with respect to the scalar product

$$(f,g) := \frac{1}{|B(0,1)|} \int_{B(0,1)} f(t) g(t) dt$$

to the system of power functions $\{x^\nu\}$, $|\nu| \le k$, located in partially lexicographic order [2].

Let $f \in L^p_{loc}(R^n)$, $1 \le p \le \infty$, $1 \le q \le \infty$,

$$m_f^k(x;r)_p := \sup \left\{ \Omega_k (f, \mathrm{B}(x,t))_p \ : \quad 0 < \ t \le r \right\} \ \ (x \in R^n,\ r > 0),$$

$$m_f^k(r)_{pq} := \begin{cases} \left\| m_f^k(\cdot;r)_p \right\|_{L^q(R^n)} & if\ 1 \le q < \infty, \\ \sup \left\{ m_f^k(x;r)_p :\ x \in R^n \right\} & if\ q = \infty. \end{cases}$$

where $\Omega_k\left(f, \mathrm{B}\left(x,r\right)\right)_p := \left|B(x,r)\right|^{-1/p}\left\|f - P_{k-1,B(x,r)}f\right\|_{L^p(B(x,r))}$ $(x \in R^n$, $r > 0)$.

Introduce the singular integral

$$S_{k,r}\left(f;K\right)(x) = \int\limits_{R^n} K_r\left(x-t\right)\left[f\left(t\right) - P_{k-1,B(x,r)}f\left(t\right)\right]dt + P_{k-1,B(x,r)}f\left(x\right),$$

where $K \in L^1\left(R^n\right)$, $K_r\left(x\right) := r^{-n}K\left(\frac{x}{r}\right)$, $r > 0$, $k \in N$, $x \in R^n$. Let $\psi\left(x\right) : = esssup\left\{\left|K\left(y\right)\right| : \quad |y| \geq |x|\right\}$, $\varphi\left(|x|\right) := \psi\left(x\right)$, $k \in N$. By Λ_k we will denote the class of all functions $K\left(x\right)$ measurable in R^n such that $\psi \in L^1\left(B\left(0,1\right)\right)$, $|x|^{k-1} \cdot \psi \in L^1\left(R^n \backslash B\left(0,1\right)\right)$. It is easy to see that $\Lambda_k \subset L^1\left(R^n\right)$.

Theorem 1. *Let $K \in \Lambda_k$, $k \in N$, $f \in L^p_{loc}\left(R^n\right)$, $1 \leq p,q \leq \infty$. Then under the convergence of the integrals in the right hand side, almost everywhere (if $q = \infty$ then everywhere) in R^n there exists finite limit $s_{k,f}\left(x\right) = \lim\limits_{r\to 0} P_{k-1,B(x,r)}f\left(x\right)$ and the following inequality is valid*

$$\left\|S_{k,r}\left(f;K\right) - s_{k,f}\right\|_{L^q(R^n)} \leq$$
$$\leq c\left(n,\psi,k\right)\left(m_f^k\left(r\right)_{pq} + \int_0^r \frac{m_f^k\left(t\right)_{pq}}{t}dt + \int_0^\infty x^{n-1}\varphi\left(x\right)m_f^k\left(4rx\right)_{pq}dx + \right.$$
$$+\int_0^r \frac{m_f^k\left(t\right)_{pq}}{t}\left(\int_0^{t/r} x^{n-1}\varphi\left(x\right)dx\right)dt +$$
$$\left. +r^{k-1}\int_r^\infty \frac{m_f^k\left(t\right)_{pq}}{t^k}\left(\int_{t/r}^\infty x^{n+k-2}\varphi\left(x\right)dx\right)dt\right),$$

where $c\left(n,\psi,k\right)$ is a positive constant dependent only on n, ψ and k.

Note that for a function $f \in L^1_{loc}\left(R^n\right)$ almost everywhere in R^n there exists the limit $s_{k,f}\left(x\right)$ and almost everywhere the equality $s_{k,f}\left(x\right) = f\left(x\right)$ is fulfilled.

Corollary 1. *Let $f \in L^p_{loc}\left(R^n\right)$, $1 \leq p,q \leq \infty$, $k \in N$, $k < \alpha + 1$, $K\left(x\right) = \left(1 + |x|^{n+\alpha}\right)^{-1}$. Then under the convergence of the integrals in the right hand side, almost everywhere (if $q = \infty$ then everywhere) in R^n there exists finite limit $s_{k,f}\left(x\right) = \lim\limits_{r\to 0} P_{k-1,B(x,r)}f\left(x\right)$ and the following inequality is valid*

$$\left\|S_{k,r}\left(f;K\right) - s_{k,f}\right\|_{L^q(R^n)} \leq c \cdot \left(\int_0^r \frac{m_f^k\left(t\right)_{pq}}{t}dt + r^\alpha\int_r^\infty \frac{m_f^k\left(t\right)_{pq}}{t^{\alpha+1}}dt\right),$$

where the constant $c > 0$ is independent of f, r, p and q.

We denote the union of all polynomials in R^n of degree at most k by P_k. Introduce the denotation $K_r * f(x) = \int_{R^n} K_r(x-t) f(t)\,dt$.

Theorem 2. *Let $K \in \Lambda_k$, $k \in N$, $f \in L^p_{loc}(R^n)$, $1 \le p, q \le \infty$, and let*

$$\forall \pi \in P_{k-1}: \quad K_r * \pi(x) \equiv \pi(x)\,(x \in R^n, \ \ r > 0)\,. \tag{1}$$

Then under the convergence of the integrals in the right hand side, almost everywhere (if $q = \infty$ then everywhere) in R^n there exists finite limit $s_{k,f}(x) = = \lim_{r \to 0} P_{k-1,B(x,r)} f(x)$ and the following inequality is valid

$$\|K_r * f - s_{k,f}\|_{L^q(R^n)} \le$$

$$\le c(n,\psi,k)\left(m^k_f(r)_{pq} + \int_0^r \frac{m^k_f(t)_{pq}}{t}dt + \int_0^\infty x^{n-1}\varphi(x)\, m^k_f(4rx)_{pq}\,dx + \right.$$

$$+ \int_0^r \frac{m^k_f(t)_{pq}}{t}\left(\int_0^{t/r} x^{n-1}\varphi(x)\,dx\right)dt +$$

$$\left. + r^{k-1}\int_r^\infty \frac{m^k_f(t)_{pq}}{t^k}\left(\int_{t/r}^\infty x^{n+k-2}\varphi(x)\,dx\right)dt\right),$$

where $c(n,\psi,k)$ is a positive constant dependent only on n, ψ and k.

It can be shown that $P_{k-1,B(x,r)} f(x) = K_r * f(x)$, where $K \in L^1(R^n)$, and condition (1) is satisfied. Thus the quantity $P_{k-1,B(x,r)} f(x)$ is expressed by a convolution type singular integral [3].

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Holomorphic continuation of integrable functions defined on the boundary of the domain

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Let $D \subset \mathbb{C}^n$ $(n > 1)$ be a bounded domain with a connected boundary of class C^1 of form

$$D = \{z \in \mathbb{C}^n : \ \rho(z) < 0\},$$

where $\rho(z)$ is a smooth function of class C^1 being real in the vicinity of the set $\overline{D}$ such that $d\rho\,|_{\partial D} \neq 0$. We identify $\mathbb{C}^n$ with $\mathbb{R}^n$ as follows: $z = (z_1, ..., z_n)$, where $z_j = x_j + iy_j$, x_j, $y_j \in \mathbb{R}$, $j = 1, ..., n$.

We consider complex straight lines $l_{z,b}$ of the form

$$l_{z,b} = \{\, \zeta \in \mathbb{C}^n : \zeta_j = z_j + b_j t, \ j = 1, 2, ..., n, \ t \in \mathbb{C}\,\} \tag{1}$$

passing through the point $z \in \mathbb{C}^n$ along the vector $b = \{b_1, ..., b_n\} \in \mathbb{C}P^{n-1}$ (the direction b is defined up to a multiplication of a complex number $\lambda \neq 0$).

Definition 1. An integrable function f on ∂D $(f \in L^p(\partial D),\ p \geq 1)$ satisfies Morera property along complex line $l_{z,b}$, if

$$\int\limits_{\partial D \cap l_{z,b}} f(z_1 + b_1 t, ..., z_n + b_n t)dt = 0. \tag{2}$$

For complex straight lines, we consider a more general condition. Let m be a fixed nonnegative integer, then the condition

$$\int\limits_{\partial D \cap l_{z,b}} f(z + bt)t^m dt = \int\limits_{\partial D \cap l_{z,b}} f(z_1 + b_1 t, ..., z_n + b_n t)t^m dt = 0 \tag{3}$$

we will call the generalized Morera property. At $m = 0$, condition (3) becomes condition (2) (see [1],[2]).

Let Γ be the germ of a C^1 manifold of real dimension $(2n-2)$. We assume that $0 \in \Gamma$ and in some neighborhood of origin the manifold Γ is of the form

$$\Gamma = \{\zeta \in \mathbb{C}^n : \Phi(\zeta) + i\psi(\zeta) = 0\},$$

where Φ, ψ are C^1 smooth, real-valued functions in the neighborhood of the point zero. Here $\zeta = (\zeta_1, ..., \zeta_n)$ and $\zeta_j = \xi_j + i\eta_j,\ \xi_j,\, \eta_j \in \mathbb{R},\, j = 1, ..., n.$

We consider complex straight lines of the form (1).It is known [1] that if for some z and for all ζ, b such that $\Gamma \cap l_{z,b} \neq \varnothing$, for $\zeta \in \partial D \cap l_{z,b}$, the function ρ defining the domain D satisfies the conditions

$$\sum_{j=1}^{n} \frac{\partial \rho}{\partial \zeta_j} b_j \neq 0, \tag{4}$$

then the curves $\partial D \cap l_{z,b}$ are smooth.

The monograph [3] presents some families of complex lines sufficient for the holomorphic extension of continuous functions. In this work, a result related to the sufficiency of a family of complex straight lines intersecting the growth C^1 of a manifold of real dimension $(2n-2)$ is obtained. The following is true

Theorem 1. *Let a domain $D \subset \mathbb{C}^n$ satisfy conditions (4) for points z lying in the neighborhood of the manifold Γ such that $\partial D \cap \Gamma = \varnothing$. Let a function $f \in L^p(\partial D)$, $(p \geq 2)$ satisfy the generalized Morera conditions* (3), *i.e.*

$$\int\limits_{\partial D \cap l_{z,b}} f(z_1 + b_1 t, ..., z_n + b_n t) t^m dt = 0$$

for each $z \in \Gamma$, $b \in \mathbb{C}P^{n-1}$ and a fixed nonnegative integer m, then the function f continues holomorphically into the domain D.

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The basis property of a trigonometric system in Morrey space

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In this study, we analyze the basis property of the trigonometric system $T = \{\cos nx,\, x\sin(n+1)\,x\}_{n=0}^{\infty}$ in Morrey space. We show that this system satisfies the Riesz property and we explore the solvability of the degenerate elliptic problem in Morrey spaces,

$$\begin{array}{rl} y^m u_{xx}+u_{yy} & =0\ , \qquad (x,y)\in(0,2\pi)\times(0,\infty)\,, \\ u\,(x,0) & =f\,(x)\,, \qquad x\in(0,2\pi)\,, \\ u\,(0,y) & =u\,(2\pi,y)\,, \qquad y\in(0,\infty)\,, \\ u_x\,(0,y) & =0\ ,\ \ y\in(0,\infty)\,, \end{array} \tag{1}$$

here $m\geq -2$ (see [1-5]).

X represents a Banach space; T_δ is a shift operator; $C_0^\infty(0,2\pi)$ denotes the space of infinitely differentiable finite functions on the interval $(0,2\pi)$; $\mathcal{L}_{p,\lambda}(0,2\pi)$ is the Morrey space of measurable functions f on $(0,2\pi)$ with the norm

$$\|f\|_{\mathcal{L}_{p,\lambda}(0,2\pi)}=\sup_{I\subset(0,2\pi)}\left(\frac{1}{|I|^\lambda}\int_I|f\,(t)|^p dt\right)^{\frac{1}{p}}<+\infty \qquad ,(0\leq\lambda<1, \qquad 1<p<\infty)\,.$$

The space $\mathcal{L}_{p,\lambda}(0,2\pi)$ is non-separable Banach space. We define a separable linear subspace of $\mathcal{L}_{p,\lambda}(0,2\pi)$ consisting of functions whose shifts are continuous in $\mathcal{L}_{p,\lambda}(0,2\pi)$ denoted by $\widehat{\mathcal{L}}_{p,\lambda}(0,2\pi)$, that is

$$\widehat{\mathcal{L}}_{p,\lambda}(0,2\pi)=\left\{f\in\mathcal{L}_{p,\lambda}(0,2\pi):\|T_\delta f-f\|_{\mathcal{L}_{p,\lambda}}\to 0,\delta\to 0\right\}.$$

This space is a Banach space with the norm of $\mathcal{L}_{p,,\lambda}(0,2\pi)$. The shift operator T_δ is defined by

$$(T_\delta f)(x)=\begin{cases} f(x+\delta) & ,\text{ if } (x+\delta)\in(0,2\pi) \\ 0 & ,\text{ if } (x+\delta)\notin(0,2\pi)\end{cases},\quad \text{for } f\in\mathcal{L}_{p,\lambda}(0,2\pi).$$

We consider the functions

$$t_0^+=1;\qquad t_n^+(x)=\cos nx;\qquad t_n^-(x)=x\sin nx;\qquad n=1,2,\ldots$$

and their biorthogonal functions

$$\varphi_0^+(x)=\frac{2\pi-x}{2\pi^2};\qquad \varphi_n^+(x)=\frac{2\pi-x}{\pi^2}\cos nx;\qquad \varphi_n^-(x)=\frac{1}{\pi}\sin nx,\qquad n\in\mathbb{N}.$$

By using the basicity criterions for the system T in Banach space, we prove that the system T forms a basis in a Banach space $\mathcal{L}_{p,\lambda}(0,2\pi)$.

For the last criteria of basisness of T, it is sufficient to prove that the projectors

$$P_{n,m}(f)=\frac{1}{2\pi^2}\varphi_0^+(g)+\sum_{k=1}^{n}\frac{1}{\pi^2}\varphi_k^+(g)\cos kx+\sum_{k=1}^{m}\frac{1}{\pi^2}\varphi_k^-(g)x\sin kx,$$

$$(\forall\,(n;m)\in\mathbb{Z}_+\times\mathbb{N}),$$

are uniformly bounded in $\widehat{\mathcal{L}}_{p,\lambda}(0,2\pi)$. Here, $g(x)=(2\pi-x)f(x), for\ x\in(0,2\pi)$.

For this aim, we use the definition of Riesz Property and (R)- space.

In order to obtain that the system T has the Riesz Property, we need to prove the following Lemma

Lemma 1. $\widehat{\mathcal{L}}_{p,\lambda}(-\pi,\pi)$ *is a (R)-space.*

And we also use the following Theorem and Proposition

Theorem 1. *The system* $S = \left\{\frac{1}{2};\cos nx;\sin nx\right\}_{n\in\mathbb{N}}$ *forms a basis in* $\widehat{\mathcal{L}}_{p,\lambda}(-\pi,\pi)$ *(*$0\leq\lambda<1$ *and* $1<p<\infty$*), [3].*

Proposition 1. *Let the system S form a basis for Banach Function Space* $X(-\pi,\pi)$*. Then the system has the Riesz property if and only if* $X(-\pi,\pi)$ *is a (R)-space.*

Considering Lemma 1, Theorem 1, and Proposition 1, we show that the following statement is true.

Statement 1. *The basis* T *in* $\widehat{\mathcal{L}}_{p,\lambda}(-\pi,\pi)$ *has the Riesz Property in this space.*

From the Lemma 1, $\widehat{\mathcal{L}}_{p,\lambda}(0,2\pi)$ is a (R)-space; from Statement, T has the Riesz Property in $\widehat{\mathcal{L}}_{p,\lambda}(0,2\pi)$. Therefore we find

$$\|P_{n,m}(f)\|_{\widehat{\mathcal{L}}_{p,\lambda}(0,2\pi)} \leq c\|f\|_{\widehat{\mathcal{L}}_{p,\lambda}(0,2\pi)}$$

Hence we give the following theorem:

Theorem 2. *The system* T *forms a basis for* $\widehat{\mathcal{L}}_{p,\lambda}(0,2\pi)$*.*

Consequently, we obtain that the system T forms a basis in non-separable Banach space $\mathcal{L}_{p,\lambda}(0,2\pi)$ and the system T has the Riesz Property.

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On the weighted integrability of the sum of series with respect to the multiplicative systems

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In this work, we consider the problem of weighted integrability of the sum of series with respect to the multiplicative systems with monotonic coefficients.

Let $\mathbb{Z}$ be the set of integers.Let $\{p_k\}_{k=1}^{\infty}$ is a sequence of natural numbers $p_k \geq 2,\ k \in \mathbb{N},\ \sup\limits_k p_k = N < \infty$. and $m_0 = 1,\ m_n = p_1 p_2 \cdots p_n,\ n \in \mathbb{N}$. For $x \in [0,1)$ with the decomposition $x = \sum_{k=1}^{\infty} \frac{x_k}{m_k},\quad x_k \in \mathbb{Z} \cap [0, p_k)$, and for $n \in \mathbb{Z}_+$ with the representation $n = \sum_{j=1}^{\infty} \alpha_j m_{j-1},\quad \alpha_j \in \mathbb{Z} \cap [0, p_j)$ the multiplicative system $\{\psi_n\}_{n=0}^{\infty}$ [1] is define by

$$\psi_n(x) = exp\left(2\pi i \sum_{j=1}^{\infty} \frac{\alpha_j x_j}{p_j}\right),\quad n \in \mathbb{Z}_+.$$

Let $1 \leq p < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$ and $\varphi(x) \geq 0$ is some locally integrable on function on $[0\,,\,1]$. By definition the function f belongs to space $L_{p,\varphi}$, if

$$\|f\|_{p,\varphi} = \left(\int_0^1 |f(x)\ \varphi(x)|^p\, dx\right)^{\frac{1}{p}} < \infty.$$

Let $\varphi(x)$ non-negative measurable on $(1,\ \infty)$ function.They say the function $\varphi(x)$ satisfies the condition E_2, if for everyone $x \geq 1$ the following inequality holds

$$\int_1^x \frac{\varphi(t)}{t} dt \leq C\varphi(x)\,.$$

Let's put

$$A_p = \sup_{0\le t\le 1}\left(\int_0^t \varphi^p(x)\,dx\right)^{\frac{1}{p}}\left(\int_t^1 (x\varphi(x))^{-p'}\,dx\right)^{\frac{1}{p'}},$$

$$B_p = \sup_{0\le t\le 1}\left(\int_t^1 \left(\frac{\varphi(x)}{x}\right)^p dx\right)^{\frac{1}{p}}\left(\int_0^t \varphi^{-p'}(x)\,dx\right)^{\frac{1}{p'}}.$$

Theorem 1. *Let* $1 < p < \infty$ *,* $\sup_n p_n = N < \infty$ *and* $\varphi(x)$ *is a non-negative, locally integrable function such that* $\max(A_p,\ B_p) < \infty$.

Then, in order for the function $f(x) \equiv \sum_{k=0}^{\infty} a_k(f)\,\psi_k(x)$*, where* $a_k(f) \downarrow 0$*,* $k \to \infty$ *belonged to the class* $L_{p,\varphi}(0,\ 1)$ *it is necessary and sufficient to satisfy the condition*

$$D_p = \sum_{n=1}^{\infty} a_n^p \cdot n^p \int_{\frac{1}{n+1}}^{\frac{1}{n}} \varphi^p(x)\,dx < \infty.$$

Theorem 2. *Let* $a_k \downarrow 0$ *at* $k \to \infty$*,* $\sup_n p_n = N < \infty$ *and* $f(x) = \sum_{k=0}^{\infty} a_k\psi_k(x)$ *and let* $\varphi(x) \ge 0$ *is measurable on* $[1,\ \infty)$ *function such that*

$$\varphi\left(\frac{1}{x}\right) \in L_1[0,\ 1],\quad \frac{1}{x}\varphi\left(\frac{1}{x}\right) \bar{\in} L(0,\ 1).$$

Then 1^0*. If*

$$\sum_{k=1}^{\infty} a_k \int_k^{\infty} \frac{\varphi(x)}{x^2}dx < \infty,$$

then

$$\varphi\left(\frac{1}{x}\right) f(x) \in L_1(0,1).$$

3^0*. If* $\varphi(x) \downarrow$ *at* $x \ge 1$*, is the positive function and*

$$\lim_{x\to\infty} \frac{1}{\varphi(x)} \int_1^x \frac{\varphi(t)}{t}dt = \infty,$$

then there is a sequence $a_k \downarrow 0$ *at* $k \to \infty$*, such that the function*

$$f(x) = \sum_{k=1}^{\infty} a_k \psi_k(x)$$

integrates on (0,1) and $\varphi\left(\frac{1}{x}\right) f(x) \in L_1(0,1)$*, however the series from* 1^0 *diverges.*

Some such results have been published in [2].In the case of trigonometric series similar problems were considered in [3].

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*HISTORY OF DEVELOPMENT OF MATHEMATICS AND MATHEMATICAL EDUCATIO*N

Conceptual features of discrete mathematics in the study of computer sciences

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Discrete Mathematics occupies an important place among the fundamental fields of mathematics and has a great influence on the development of Computer Science. Its significance lies in the fact that it provides basic tools and concepts for analyzing, modeling, and solving a wide range of problems related to computer technology and information systems. The study examines the significance of Discrete Mathematics for computer technology and science, in general. It is a key tool for developing algorithms, data structures, optimization methods, cryptography, information theory, computational theory, and many other fields.

In our republic and foreign countries, scientific research is being conducted to study the importance of "Discrete Mathematics" in teaching Computer Science. In particular, the scientific research at universities conducted by Cécile Ouvrier-Buffet, Antoine Meyer, Simon Modeste was devoted to the study of the interaction of Discrete Mathematics, Arithmetic and Computer Science [1]. Rafael Pass focused his studies on the content of the course "Discrete Mathematics" in the field of Computer Science [2]. Discrete Mathematics is included in the course block in mathematical and natural sciences. As a result of developing skills in knowing the basic concepts of this subject and performing operations, students can master in "Data Structure and Algorithms", Programming" specified in the state educational standards, "Process Research", "Mathematical modeling", "Intelligent systems", "Computer organization", "Design of digital logic devices", "Inlined computer systems", "Information security", "Mobile communication systems". Students acquire basic knowledge that forms the methodological foundation and "Discrete mathematics" serves as the basis for these disciplines in their development [3].

A review of scientific publications revealed a wide range of concepts in Discrete Mathematics that have a significant impact on computer science. The basic concepts of discrete mathematics and the relationship between these concepts are shown in Fig. 1.

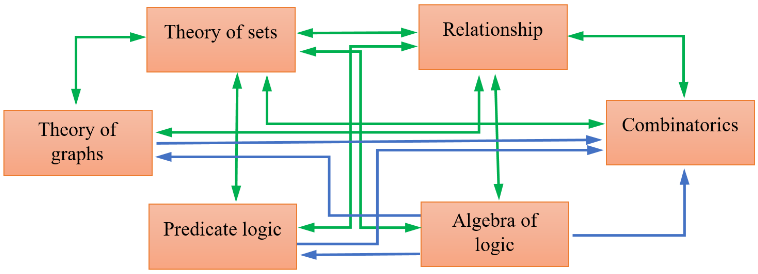

Figure 3: Basic concepts of Discrete Mathematics and the relationship between these concepts

Studying these concepts of Discrete Mathematics allows computer specialists to develop more efficient algorithms, improve software performance, ensure information security, and create new data analysis techniques. Thus, Discrete Mathematics plays an important role in developing Computer Science and related technologies.

Logic algebra has its place in various areas of computer science. Its rigor and formality provide the basis for developing algorithms, programming, artificial intelligence systems, and many other fields where exact and formal thinking is required. Here are some ways logic influences Computer Science: *algorithm design, programming, artificial intelligence, formal methods of analysis and verification, computability theory, and mathematical logic.*

Graphs are abstract mathematical structures that consist of vertices (nodes) and edges (links) between those vertices. They play a key role in Computer Science and Information Science because of their versatility and applicability to a wide range of problems. Ways to use graphs in computer science are *network modeling, shortest path algorithms, route optimization, graph databases, social network analysis, software compilation and analysis.*

Combinatorics is an important branch of Discrete Mathematics that has wide applications in computer science. Its concepts and methods are used in solving various problems related to information processing and algorithm development. Here are a few ways combinatorics influences computer science: *sorting algorithms, search algorithms, data encryption, data encoding, genetic algorithms, resource allocation, and scheduling.*

Indeed, Discrete Mathematics plays a key role in developing modern computer science and technology. The fundamental concepts of this branch of mathematics provide computer scientists and engineers with tools and a fundamental understanding of the principles underlying various aspects of computer science and computer technology. Here are some of the main aspects in which Discrete Mathematics plays a significant role: *modeling and analysis, algorithm development, optimization, security and cryptography, artificial intelligence, and machine learning.*

Thus, understanding the fundamentals of Discrete Mathematics is a necessary foundation for successful development in the field of computer technology. It not only provides tools for solving complex Computer Science problems but also teaches abstract and logical thinking, which is a key skill in the IT field. Without knowledge of Discrete Mathematics, it would be impossible to create effective algorithms, develop reliable security systems, analyze large amounts of data, or even create computer games and applications. In the modern world, where computer technology penetrates all spheres of life, understanding Discrete Mathematics becomes the key to a successful career and innovative development.

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Using symmetry in solving problems with a parameter

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The group of problems with a parameter includes those in which it is necessary to establish the values of the parameter at which the equation (system or inequality) has a "single solution", "an even number of solutions" or "an odd number of solutions". Almost always, such problems have a characteristic feature: their conditions do not change either when the sign of one or several variables is replaced with the opposite one ("symmetry" with respect to the sign), or when several variables are rearranged ("symmetry" with respect to the rearrangement of variables). When solving problems of this kind, the following procedure is used: first, a symmetry check is performed; secondly, by checking the fulfillment of necessary conditions, acceptable values of the parameter are found (with "symmetry" with respect to the sign of a variable, its zero value is substituted; with "symmetry" with respect to the permutation of variables, all variables are denoted by one letter); thirdly, the sufficiency of the conditions is checked, i.e. for the found acceptable values of the parameter, a check is made to ensure that, given the obtained values of the parameter, the equation (system, etc.) actually has the required number of solutions.

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On the influence of Tusi's works on European mathematical thought

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The role of Tusi's works in the formation of trigonometry as an independent science and their influence on Regiomontanus, who in Western science is considered its founder, is generally recognized. Mathematicians in Europe from the 14th century knew the Latin translation of Nasireddin Tusi's Exposition of Euclid, as revised by his son and student Shirazi, now called "Pseudo-Tusi". This translation was used by mathematicians, especially in the matter of the "fifth postulate". Wide attention is now being drawn to the so-called "Tusi pair" - lemma, proven by Tusi and very likely used by Copernicus. All these facts are reflected in numerous literature.

The above facts are confirmed by sources and are generally accepted. We will give several hypothetical versions of the influence of Tusi's work on the mathematical thought of Europe. Note that the latest versions arose as a result of our research into his works. Note that the latest versions arose as a result of our research into his works.

In 1228, the Italian traveler, merchant and mathematician from Pisa Leonard of Pisa (Fibonacci) (1188 - 1260) "published" the fundamental work "The Book of Abacus" (edited version), where he collected in the most complete way the works of Arabic-speaking scientists on arithmetic. The work of Tusi, (which was called by contemporaries as the "Shah of Shahs of Science"), "Collection of arithmetic using a board and dust," which was written in 1227, could not help but attract the attention of Leonard of Pisa. Note that by "abacus" hi meants a table covered with dust in the east. In this treatise, Tusi points out the insufficiency of checking calculations using the "measurement" method, which was widely used and valuable as a result of calculations on the board, that is. using "abacus" when intermediate calculations were not saved. Beginning in the 13th century, the University of Pisa carried out an intensive translation of Arabic manuscripts into Latin. in the 15th century, the Italian mathematicians Paciolli and Schuquet became interested in the works of Leonardo of Pisa, especially in connection with the accounting methods laid down by him. The indication of the inadequacy of the "measures" method in the history of mathematics is attributed specifically to them but is well known. that they used the "Book of Abacus."

In the "Clavis matematica" (Key to Mathematics) by one of the creators of mathematical symbolism, William Outhred (1574-1660), an additive method of denoting degrees is given, which, up to the replacement of Arabic letters with Latin, coincides with the principle that was proposed by Tusi in his "Collection", and perhaps in some other manuscript. Note that the use of the notation is not known either among Arab mathematicians, Tusi's predecessors, or among mathematicians of a later period. It is also known that mathematicians in England of the 15th and 16th centuries had Arabic manuscripts and their Latin translations. Outhred's acquaintance with Arabic sources is also indicated by the themes of his works, in particular, about decimal fractions with which Europeans became acquainted from J. Kashi's treatise "The Key to Arithmetic". Tusi's definition of a real positive number, formulated in the "Collection", was the completion of the attempts of prominent mathematicians of the East, including Omar Khayyam, to define numbers through relations. This definition coincides with the definition of number declaratively given. Isaac Newton, although Newton's acquaintance with Tusi's work is not established. However, Newton highly valued the work of Wallis, who was at least familiar with the Latin translation of the "Exposition of Euclid". In particular, Wallis wrote in his comments: "I will finally accept the nature of the relation and the doctrine of similar figures." He also introduces an axiom about the existence of similar figures and publishes Tusi's proof of the fifth postulate in his lectures. His scientific connection with Descartes who interpreted the relation geometrically as did Euclid, Tusi, and his predecessors is known. The German mathematician Pasch (1843-1930) formulated an axiom, which, in fact, is an intuitive statement in Tusi's proof of the 5th postulate (formerly Nasawi) that deprives it of rigor. This, in particular, leads him to the idea of the need to formalize geometry. In addition, he formulates the axiom of belonging, which Tusi cites in his "Exposition".

Some remarks on modal proposals in Tusi's treatise "Tajrid al-mantik"

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In Tusi's treatise "Tajrid al-mantik" the classification of proposals, in particular modalities, is considered in chapter 3. Syllogisms compiled from modal and also non-modal proposals are studied in Chapter 4. Usually, in classical works alethic and temporal modality are considered. Depending on the relationship between the subject and the predicate three types of modality are defined: absolutely, necessaries, possibilities or impossibilities. It is stated two ways for classification of modality:

1. Proposal in which modality does not noted is called absolutely proposal. If verdict about the subject by predicat is necessary then it is called necessary proposal. If this verdict is only possible then it is called potential proposal.

2. The verdict may be factically is valid: if it is non-nessecery it is called absolutely, in the case necessary it is called necessary, and if in fact it is non valid, but may be realized –it is called potential proposal.

Althouh Tusi noted both ways, he followed the first way.

In addition, in the modalities, depending on how continues verdict - permanently or no permanently, also conditions on the subject the propositions may have different qualities. Typically, the modalities of the propositions are fixed after verdict: "A is B -absolutely", "A is B -necessarily", "A is B – permanently (always)", "A is B – no permanently" so on.

Absoluties: the verdict is realized really in some moment. This realization may be permanently, or non-permanently. **Absoluties in general sence** mean that the verdict may be permanently, and also non-permanently, but permanentness of the contradiction of the verdict, i.e. non realization of the assertion always, is impossible. **Absolutness** in particular sence – in this case the verdict is not permanent: for example, the proposal "A is B" means that "A is B - non permanently".

Possibility – description of the subject by predicate potentially is possible. **General Possibility** – in parallel with the given statement, its contraction is also possible. **Particular possibility** - in this case contradiction of the assertion is impossible.

Descriptive proposal – subject descripted by some properties.

Temporal proposals – the realization of verdict connected by some time. This time may be concrete, or non-concrete. If subject does not descripted by any property, then the verdict is about substation of subject.

Deniel of permanentness is absolutely in general sence. For example, deniel of the statement “every A is B - permanenetly” is the statement “some A is not B” absolutely in general sence. Deniel of “A is B” is “A is not B - always”. It is impossible. Deniel of the possibility is deniel of necessarity. For example “A is B – by posibility” contradiction is “A is not B – does not necessary”

Permanentness is more general than necessarity, possibility is more general than absoluteness, general possibility is more general than general absoluteness.

Also, existence or generality quantifiers may take part in such propositions. In this case, it is important to understand how and on which one the quantifier influences? For example, the proposition “some A-s is B - permanently” can be interpreted in different ways:

- Some invariant and unchanged A-s are B always;

- Always there is any A (may be different A’s for different times) which is B.

Studying of contradictions of modal propositions and results of logical figures show that quantifiers influence the unchanging or invariant elements of the subject only.

Therefore,

- the proposition “some A-s is B - permanently” must be understand in following way: there are some concretely and unchanged elements of A, which are B every times.

- the proposition “some A-s is B - permanently” must be understood in following way: there are some concretely and unchanged elements of A, which are B every times. Contradiction of this proposition consist of two propositions – “every A is B - permanently” or “every A is not B - permanently”;

- The proposition “arbitrary A is B - permanently” means that, each element of A is B, all times;

- the proposition “some A is B – non-permanently” means that there are some unchanging elements of A, which are B only sometimes;

The place of mathematical analysis in the preparation of mathematics teachers

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The object of mathematical analysis includes: variable quantities, functions with one or more variables, their limits, differentials, integrals, etc., which are fundamental concepts. These concepts form the basis of mathematics in secondary education. Therefore, it is important to ensure that future teachers have a broad and substantial understanding of functions and their presentation methods. Students receive information about the definition of a function and methods of presentation in the first lessons of Mathematical Analysis. Although they may encounter alternative methods of presenting functions in subsequent lessons, they mostly recall that functions are presented analytically, graphically, in tables, and verbally in the first lessons. However, unlike high school graduates, they should have sufficient knowledge about functions being presented in sequences, integrals, differential equations, as well as in non-obvious ways. Mathematical analysis provides sufficient opportunities for the realization of these tasks.

The effectiveness of teaching mathematics at any level in pedagogical universities lies in the comprehensive nature of its explanation. It contributes to the development of creativity in students and helps them to select the most appropriate approach among various perspectives on the subject. This is an essential issue today and serves the development of the creativity and methodological preparation of mathematics teachers.

Both in secondary and higher education, diverse perspectives on each mathematical issue and learning to prove a theorem through different methods help students to internalize the subject better. It is more beneficial and instructive to solve one example in two different ways than to solve two examples using the same method.

In this regard, the authors present their ideas on the teaching of mathematical analysis in universities.

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A method for reducing a higher order definite integral of the function

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In the paper, we give a technique for reducing the n-th order definite integral of the function to a lower-order definite integral.

Theorem 1. *Asuume that the function* $f(x)$ *in the interval* $[a, b]$ *is a Rieman integrable function and retains its sign in the interval* $[a, b]$*. Then for any number* n $(n \geq 2)$ the formula

$$\int_a^b f^n(x)dx = \mu^{n-1} \int_a^b f(x)dx, \tag{1}$$

is valid, here the number μ *is a definite number belonging to the interval* $[m, M]$.

Proof. For definiteness $0 \leq m \leq f(x) \leq M$, $x \in [a, b]$. For $n = 2$ this fact is valid. Let us prove the theorem for the case $n = 3$.

$$m \leq f(x) \leq M$$

For

$$\int_a^b f^3(x)dx = \mu_1 \cdot \mu_2 \int_a^b f(x)dx, \tag{2}$$

here $m \le \mu_1, \mu_2 \le M$, is a number satisfying the conditions. For definiteness assume that $\mu_1 \le \mu_2$, then $\mu_1^2 \le \mu_1, \mu_2 \le \mu_2^2$; we take $\mu = \sqrt{\mu_1 \mu_2}$ and it follows from these inequalities that

$$\mu_1 \le \sqrt{\mu_1 \mu_2} = \mu \le \mu_2.$$

Thus, for $n = 3$ we proved that there exists such a number μ $(0 \le m \le \mu \le M)$ that the equality

$$\int_a^b f^3(x)dx = \mu^2 \int_a^b f(x)dx \tag{3}$$

is satisfied. By the mathematical induction method we prove the case $n \ge 3$.

The theorem is proved.

Theorem 2. *Assume that the function $f(x)$ is a Reimann integrable function, in the interval $[a, b]$ and $A \le f^2(x) \le B$, $x \in [a, b]$. Then for $k \ge 2$ the formula*

$$\int_a^b f^{2k}(x)dx = \mu^{k-1} \int_a^b f^2(x)dx \tag{4}$$

is valid, here μ is a number satisfying the conditioins $(A \le \mu \le B)$

Proof. Taking $g(x) = f^{2(k-1)}(x)$ for this function we can write

$$\int_a^b f^{2k}(x)dx = \int_a^b f^2(x) \cdot f^{2(k-1)}(x)dx = \mu_1 \cdot \int_a^b f^2(x)dx, \tag{5}$$

here μ_1 is a definite function satisfying the conditions$(A \le \mu_1 \le B)$.

Again

$$\int_a^b f^{2(k-1)}(x)dx = \int_a^b f^2(x) \cdot f^{2(k-2)}(x)dx = \mu_2 \cdot \int_a^b f^{2(k-2)}(x)dx.$$

Continuing this process

$$\int_a^b f^{2k}(x)dx = \mu_1 \cdot \mu_2 \cdot ... \cdot \mu_{k-1} \cdot \int_a^b f^2(x)dx, \tag{6}$$

here $A \le \mu_i \le B$ are definite numbers satisfying the conditiouns μ $(A \le \mu \le B)$. It is clear that in this case, we can find such a number

$$\mu^{k=1} = \mu_1 \cdot \mu_2 \cdot ... \cdot \mu_{k-1} \tag{7}$$

that. Taking into account equality (7) in (6), we obtain formula (4).

The theorem is proved.

Theorem 3. *Let the function $f(x)$ be a Reimann integrable function in the interval $[a,b]$ and $f^2(x)$ be a continous function in the interval $[a,b]$. Then for $k \geq 2$ the formula*

$$\int_a^b f^{2k}(x)dx = f^{2(k-1)(c)} \int_a^b f^2(x)dx \tag{8}$$

is valid. Here c $(a \leq c \leq b)$ is a definite point.

Theorem 4. *Assume that the function,$f(x)$ is a continuous function in the interval $[a,b]$. Then for $k \geq 2$ the formula*

$$\int_a^b f^{2k+1}(x)dx = f(c_1) \cdot f^{k-1}(c_2) \int_a^b f^2(x)dx, \tag{9}$$

is valid, where c_1 and c_2 are definite numbers satisying the condiitons $(a \leq c_1, c_2 \leq b)$.

A methodology for studying the basic properties of an ellipse

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1. Derivation of the equation of the ellipse.

We are given a rectangular Cartesian coordinate system XOY. On the abscissa axis we mark two points $F_1(-c,0)$ and $F_2(c,0)$, where c $(c>0)$ is a some positive number. The points F_1 and F_2 are called focal points of the ellipse.

Let $\mu(x,y)$ be an arbitrary point on this coordinate system XOY. Denote by r_1 and r_2 the distance from the point $M(x,y)$ to the points F_1 and F_2 respectively, i.e.

$$r_1 = \sqrt{(x+c)^2+y^2},\ r_2 = \sqrt{(x-c)^2+y^2}. \tag{1}$$

Let a $(a > c)$ be an arbitrarily fixed positive number.

On the plane XOY we find a geometrical locus (i.e. an equation) of the points $M(x, y)$, whose sum of distances from the given two points $F_1(-c, 0)$ and $F_2(c, 0)$ is a constant number 20, i.e.

$$r_1 + r_2 = 2a. \tag{2}$$

Definition. The above stated geometric locus of point is called an ellipse.

By squaring both sides of (2), we obtain

$$2r_1r_2 = 4a^2 - r_1^2 - r_2^2.$$

Consequently, using (1), we have

$$r_1r_2 = 4a^2 - E^2 - C^2 - A^2.$$

Hence, squaring again, we obtain

$$(a^2 - c^2)x^2 + a^2y^2 = a^2(a^2 - c^2).$$

Consequently,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \tag{3}$$

where $b^2 = a^2 - c^2$.

Thus, on the plane XOY the geometric locus of the points $M(x, y)$, whose sum of distances from the given two points $F_1(-c, 0)$ and $F_2(c, 0)$ is a constant number $2a$, determined by the equation (3).

(3) is called a canonical equation of the ellipse and the numbers a and b are called major and minor semi-axes, respectively, and the number $2c$ is called a focal distance.

We have the following theorem

Theorem 1. *Let a real continuous function on the segment and at each point of the interval* (a, b) *have a finite derivative* $f'(x)$ *and be Riemann integrable in the segment* $[a, b]$.

Then for the length of the curve Γ, having described the graph of the formula $y = f(x)$ for $a \leq x \leq b$ we have the following formula

$$\partial\Gamma = \int_a^b \sqrt{1 + (f'(x))^2}dx. \tag{4}$$

2. A formula for calculating the length of the ellipse and its main features

We have

Theorem 2. *For the length of the ellipse determined by the formula (3), we have the formula*

$$l = d \cdot \varphi(e), \tag{5}$$

where

$$\varphi(e) = 2\int_0^1 \sqrt{\frac{1-e^2t^2}{1-t^2}}dt, \tag{6}$$

$e = \frac{c}{a}$ *is called the eccentricity of the ellipse and is always* $0 < e < 1$, l *is the length of the ellipse.*

Theorem 3. *If focal points* $F_1(-c, 0)$ *and* $F_2(c, 0)$ *are marked, then* $\lim\limits_{c\to+0} \frac{l}{2a} = \pi$, *where* l*is the length of the ellipse.*

Theorem 4. *If "a" is marked, then* $\lim\limits_{c\to+0} \frac{l}{2a} = \pi$, *where* l *is the length of the ellipse.*

Theorem 5. *The function* $\varphi(e)$, $0 \le e \le 1$, *is a strictly decreasing function and its values decrease from* π *to 2 and* $\lim\limits_{c\to a-0} \frac{l}{2a} = \varphi(2) = 2$.

Theorem 6. *For all the ellipses with identical eccentricities the ratio of the length of the ellipse to its diameter is a constant number*

$$\frac{l_1}{d_1} = \frac{l_2}{d_2}.$$

A methodology for applying geometry to geodesy problem

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Problem 1. Calculate the distance from the accessible point A to some point B visible from A.

Solution. We take any fixed point C 1) $C \notin AB$, 2) C is an accessible point, 3) the point A and point B are visible from C.
1) We determine the distance $AC = a$.
2) Determine the angle α.
3) Determine the angle γ (see fig .1).

Based on the fact that the sum of the angles of the triangle is 180^0, we obtain

$$\beta = 180^0 - (\alpha + \gamma).$$

Now , by the theorem of the sines we obtain :

$$\frac{\sin\beta}{AC} = \frac{\sin\gamma}{AB}.$$

Hence

$$AB = \frac{\sin\gamma}{\sin\beta} \cdot a. \tag{1}$$

The required distance will be determined by formula (1).

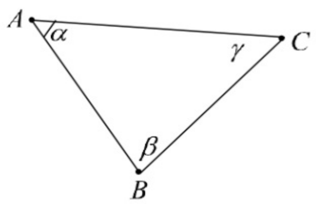

Fig.1.

Problem 2. Let us determine the distance between inaccessible points A and B.

Solution. We take some accessible point C, from which the points A and B are visible. We determine the angle γ. By the problem 1 we determine the distance AC and BC. Then, by the theorem of the cosines

$$AB^2 = AC^2 + BC^2 - 2AC \cdot BC \cdot \cos\gamma, i.e.$$

$$a^2 = AC^2 + BC^2 - 2AC \cdot BC \cdot \cos\gamma. \tag{2}$$

Thus, the distance between the inaccessible points A and B is determined by formula (2).

Problem 3. Find the area of the triangle ABC for which all the vertices of the triangle are inaccessible points.

Solution. By problem 2 we determine the distance between the inaccessible points $AB = c, BC = a, AC = b$. Now, by the Heron formula, we calculate the required area

$$S = \sqrt{p(p-a)(p-b)(p-c)},$$

where $p = \frac{a+b+c}{2}$.

Problem 4. Four inaccessible points A, B, C and D are randomly marked. Does the point D fall into the ΔABC?

Solution. Using problems 1-3, we find the area of the triangles.

S is the area of ΔABC,

S_1 is the area of ΔABD,

S_2 is the area of ΔADC,

S_3 is the area of ΔBDC.

If the equality $S = S_1 + S_2 + S_3$ is valid, the point D falls into the triangle ABC. Otherwise ,the point D does not fall into the triangle ABC.

Problem 5. We are given inaccessible points A and B. Find the midpoint of the segment $[AB]$.

Solution. We take such an accessible point C, that $M_1 = \frac{1}{2}AC_1$ (i.e. $M_1 \in [AC_1]$ and $M_1A = M_1C_1$), N_1 an accessible point $N_1 = \frac{1}{2}C_1B$ and N_1 is an accessible segment, M_1N_1 $P_1 = \frac{1}{2}M_1N_1$. In a similar way we construct $C_2, M_2,$ N_2, P_2 (Fig.2).

The point of intersection of the rays C_1P_1 and C_2P_2 will be the half of the segment AB.

Problem 6. We are given an inaccessible triangle ABC. We determine triangle $\alpha = A$.

Solution. By the above method, we determine the lengths of inaccessible segments AB, BC and AC.

By the theorem of the cosines, we have

$$BC^2 = AB^2 + AC^2 - 2AB \cdot AC \cdot cos\ \alpha.$$

Here

$$\cos\alpha = \frac{AB^2 + AC^2 - BC^2}{2AB \cdot AC}.$$

Consequently,

$$\alpha = \arccos\frac{AB^2 + AC^2 - BC^2}{2AB \cdot AC}.$$

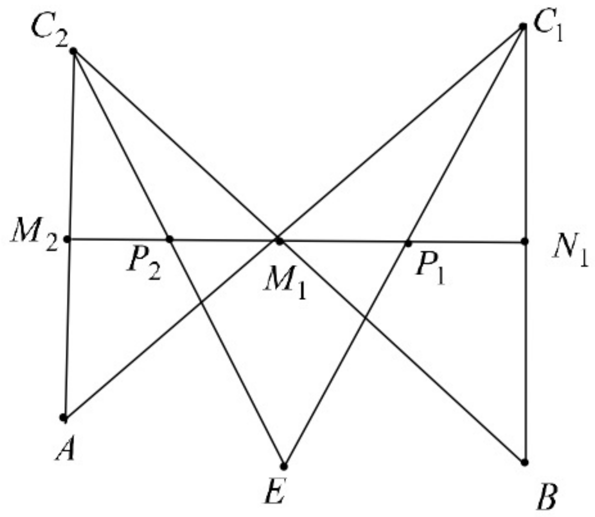

Fig.2.

On the problems of justification of mathematics

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The development of research on the foundations of mathematics in the 19th-20th centuries went through, without going into detail, two stages. The first stage led to the creation of several equivalent theories of real numbers, the definition of the basic concepts of analysis on this basis, and the construction of a descriptive theory of sets and functions. The second stage consists of identifying and studying the logical means used in constructing a mathematical theory, in particular, analysis. Let us note this division and turn to the first stage.

Historically, it turned out to be a construction of "classical mathematical analysis". The foundation of this version of analysis is the set of all real numbers, ultimately defined as the infinite sets of rational numbers; rational, which means, after all, natural numbers. Note that this approach requires fluency in the concept of actual infinity. This means that it is fundamentally opposed to the approach of the Greek geometers: what was an undesirable paradox for the Greeks is often a useful construction for 19th-century mathematicians.

Note that the attractive feature of actual infinity is its logical simplicity. It is easier to define an infinite set than a finite one.

For CityplaceBolzano, geometric concepts are too complex to be original. Dedekind quite consciously strives to build a "kingdom of numbers" independent of ideas about measurable quantities, in particular, from space-time intuition. For Frege and Russell, natural numbers are too complex. This means that they themselves need interpretations and such interpretations must be sought in the area of classes and relations or sets.

This is the second remarkable feature of the direction in which the justification of the analysis moved: not only actually infinite sets receive citizenship rights, but in general the concept of a set (without taking into account its finiteness or infinity) is perceived as more fundamental and simpler than the concept of a natural number. Although this view was opposed at the beginning of the 20th century. Since strong objections have been expressed, not only by intuitionists, the emergence of such a view is not accidental. Its roots can be traced back quite far, although only in Frege and Russell does it receive quite adequate expression.

With the entry into the sphere of sets, mathematics received a field that is a universal supplier of material for the interpretation of mathematical concepts. Now we can give some, albeit rather approximate, characteristics of the first stage of development of research on the foundation of mathematics: this is the development of a logistical, i.e., set-theoretic concept of mathematics

The creation of ""Principia" mathematica" by A. Whitehead and B. Russell was the logical completion of the first stage and the beginning of the second. Regarding the first stage, it should also be noted that the form that the ideas of Dedekind and Cantor acquired in Principia differ very significantly from the original form of their expression, which has since been called "naive". On the other hand, by the time of this grandiose summing up, it had become quite plausible that everything in the direction was only one of many possible ones. One of the first was the system of predicative analysis by G. Weyl (1918), which was generally close to Principia, but rejected the axiom of reducibility. Then L. Brouwer's intuitionistic analysis appeared. Brouwer tries to return the natural number to its dominant role in mathematics and, on the other hand, receives an analysis of a completely different style, which is not a subsystem of the classical one. The construction of non-classical systems is proceeding at an ever-accelerating pace, and we will not follow this process. Let us only note that, apparently, the main reason for the low popularity of intuitionism outside of works related to the foundations of mathematics is that its logic is too complex.

But we also raised another question - about the significance of works related to the justification of mathematics for mathematics itself. Here you should pay attention to the following. Of course, justification as such does not lead directly to the discovery of new facts. But in the process of justification, first of all, methods can be created that then acquire independent value. An example is Cantor's concept of functional sequence. Introduced initially to justify the concept of a real number, it then became an extremely popular tool in topology, functional analysis, and even in the theory of algebraic numbers and constructive analysis. Justification is always an analysis of the initial theory and inevitably changes it, opening new horizons.

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Formation of students divergent thinking and divergent skills in the mathematics teaching process of grades V-IX

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The article examines the problem of forming divergent thinking and divergent skills of students in the process of mathematics training in V-IX grades of general education schools. It is known that at the current stage of education development, finding different ways to solve the problem, comparing possible options of activities, and forming the ability to make optimal decisions in the conditions of multiple choices are urgent issues. The State Education Standards for secondary general education emphasize not only the implementation of meaningful educational activities, but also the importance of creating and organizing training conditions aimed at understanding objective reality, forming multifaceted perceptions, and the ability to see different ways of solving problems.

In order to form these skills in students, high-quality training aimed at the development of a special type of thinking characterized by variability, variation, originality and non-standardity is needed during school education [1].

First of all, mathematics lessons have a significant potential for the development of such thinking, and especially in the teacher's teaching practice, various mathematical problems are used, that is, problems that have several solutions, different interpretations of given conditions, or problems that give the correct answer in several options [4, p. 11].

In the scientific literature, we find two components of thinking - ***convergent*** and ***divergent*** forms of thinking in the cognitive taxonomy created by G. Gilford in 1967. The problem of the formation of divergent thinking was widely considered in the works of J. Gilford, who developed a theory about these types of thinking. The signs of divergent thinking are defined as ***flexibility, fluidity*** and ***originality*** [3, p. 434]. D. B. Bogoyavlenskaya concept of "divergence" in psychology is associated with the ability to create [2, p. 59].

After defining the convergent and divergent components of thinking, the terms "convergent issue" and "divergent issue" appeared in the scientific literature. The first type of problems indicates the existence of problems that can be solved by one method, and the second type of problems indicates the existence of problems that can be solved by two or more methods. However, since in everyday practical activity a person is most often faced with the second type of problem, the ability to solve such problems is an important component of school education: in real life and in future professional activities, students will have to find answers to questions related to various topics in the most diverse situations with various conditions.

Despite the fact that the problem of using divergent problems in the process of education of schoolchildren is actively studied in the scientific literature, in the practice of mathematics education, convergent problems are used more often, the answer to which can be found by using relevant laws, rules, algorithms, formulas, theorems, etc. That is why one of the tasks facing the mathematics teacher is to develop divergent issues and include them in the content of the lessons. These issues are a means of developing students' creative thinking and creative skills, and also help graduates to better prepare for final exams and competitive tests.

In the course of school mathematics, we encounter several types of divergent problems.

S. M. Krachkovsky classifies them, for example, as follows [4, p. 18].

Indeed, divergent problems can act as an effective tool for preparing schoolchildren for the state exam in mathematics, because this exam aims to apply the complex of knowledge acquired by secondary school students, to check the formation of the ability to see and find different ways, to solve problems, to show the ability to creatively approach the solution. Divergent issues in mathematics classes allow to present various topics in interaction and contribute to the deepening and systematization of knowledge.

A large number of standard problems solved with the help of a quadratic equation can be considered a divergent problem since the quadratic equation can have several solutions when solved. Mastering each of them is important for students studying at the secondary general education level.

Problem 1. Find the smallest root of the equation: $x^2 - 3x - 18 = 0$.

Let us note the following methods for solving the given quadratic equation.

Method I. Finding the solution of the quadratic equation by discriminant

$$D = b^2 - 4ac = (-3)^2 - 4 \cdot 1 \cdot (-18) = 9 + 72 = 81 > 0.$$

$x_1 = \frac{-b+\sqrt{D}}{2a} = \frac{3+\sqrt{81}}{2} = 6; x_2 = \frac{-b-\sqrt{D}}{2a} = \frac{3-\sqrt{81}}{2} = -3$. Answer: -3.

Method II . Solving the quadratic equation by separating its perfect square.

$$x^2 - 3x - 18 = x^2 - 2 \cdot 1,5x - 2,25 - 20,25 = 0 \Leftrightarrow$$

$$\Leftrightarrow (x - 1,5)^2 - 4,5^2 = 0 \Leftrightarrow (x - 1,5 - 4,5)(x - 1,5 + 4,5) = 0 \Leftrightarrow$$

$\Leftrightarrow (x-6)(x+3) = 0 \Rightarrow \left[\begin{array}{c} x = 6, \\ x = -3. \end{array} \right.$ Answer: -3.

Method III. Solving the quadratic equation by separating the quadratic trinomial into multipliers.

$$x^2 - 3x - 18 = 0 \Leftrightarrow x^2 - 6x + 3x - 18 = 0 \Leftrightarrow$$

$\Leftrightarrow x(x-6) + 3(x-6) = 0 \Leftrightarrow (x-6)(x-3) = 0 \Rightarrow$

$\Rightarrow \left[\begin{array}{c} x = 6, \\ x = -3. \end{array} \right.$ Answer: -3.

Method IV. Solving the quadratic equation by using the Viet's theorem

Since for the given quadratic equation $a = 1$:

$x_1 + x_2 = 3, \; x_1 x_2 = -18 \Rightarrow \left[\begin{array}{c} x = 6, \\ x = -3. \end{array} \right.$ Answer: -3.

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Application of Routh's theorem to high school problems

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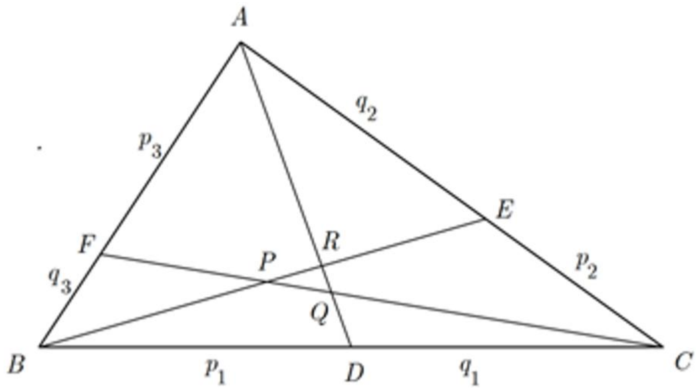
With the development of ICT education in our country, there are some theorems, axioms, etc. that continue to be the focus of attention of teachers, as well as some issues that attract special attention from high school students. In the curriculum system, it is necessary to consider some named theorems related to foreign literature. Stewart's theorem is one of the theorems that makes it easier to learn a particular language. Although university students with mathematical knowledge are familiar with theorems only in the higher mathematics course at the university, they also study these theorems, lemmas, etc. at school. Therefore, students' familiarity with theorems over time is, in a sense, one of the important factors in creating in their subconscious a certain order of these theorems or lemmas. For example, although it is not as popular a theorem in high school as, say, the Pythagorean Theorem, the cosine theorem does not receive much attention. In addition, in some articles, we can familiarize ourselves with Stewart's theorem. The situation is similar with the ranking of students according to "so-called theorems" in the curriculum abroad. For example: We can recognize Routh's theorem, which we can talk about with confidence because it has not been mentioned anywhere until now. The purpose of this study is to impart knowledge of theorems to secondary school students. Let's consider a brief summary of Routh's theorem.



$$\frac{S_{\Delta PQR}}{S_{\Delta ABC}} = \frac{(p_1p_2p_3 - q_1q_2q_3)^2}{(p_1p_2 + q_1q_2 + p_1q_2)(p_2p_3 + q_2q_3 + p_2q_3)(p_3p_1 + q_3q_1 + p_3q_1)}$$

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Comments by N. Tusi on the work of Archimedes "On the sphere and the cylinder"

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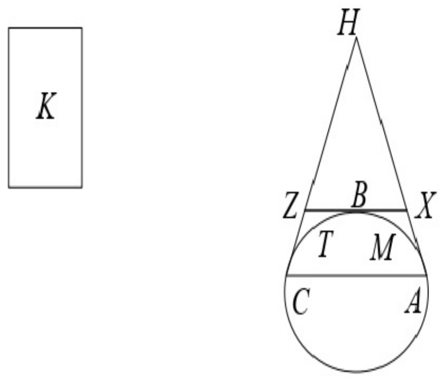
Theorem 1. *If on the lateral surface of a straight cylinder, there are two straight lines reaching the bases of the cylinder, and tangents and pairs intersecting are drawn from the end of these straight lines to the circles of the bases, then the surface between the tangents and two straight lines on the lateral surface of the cylinder will be greater than the surface of the cylinder between the two straight lines located on the side surface of the cylinder.*

Let one of the bases of a straight cylinder be circle ABC (Fig. 1).

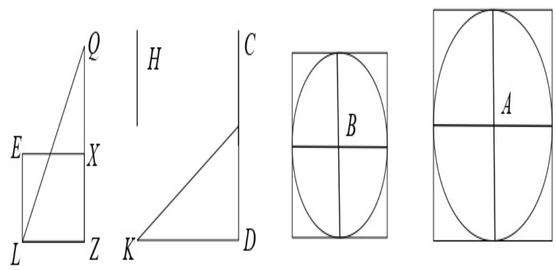


N. Tusi makes the most important remark to this Theorem 15, contained in Theorem 11: "Under the assumptions of Archimedes: "Of surfaces that have a common boundary located on a plane, the plane will be the smallest. Two surfaces that have a common boundary located on a plane will always be unequal if both of them are convex in the same direction and one of them is completely surrounded by the other surface and the plane containing their common boundary, while the volumetric surface will be smaller."

"The statement of the theorem was used in the proof. This is repeated several times in the text. This is an unacceptable contradiction of Archimedes."

Theorem 2. *The surface of any straight cylinder minus the bases is equal to a circle with a radius that is the average proportional between the edge of the cylinder and the diameter of its base.*

Let circle A be the base of a right cylinder (Fig. 2).



Triangle KDS will be equal to the figure drawn outside circle A: S KDS=Sopis.A. Because its base is equal to the perimeter of this figure, and its height is equal to the radius of circle A. The figures described around circles A and B are similar, and the ratio of the figure described around circle A to the figure described around circle B is equal to the ratio of the squares of the radii of these circles:

$SKDS\ S$ description $B = R_A^2\ R_B^2 = R_A^2 \cdot DK\ H^2 \cdot ZL = SKSD\ QZ \cdot ZL = SKSD\S LZQ$

S description $B=\ S\ LQZ$

$S\ LQZ \backslash S$ inscription $B{=}S$ description $B \backslash S$ inscription B less inv. cyl.$B \backslash S_{kpð} B$

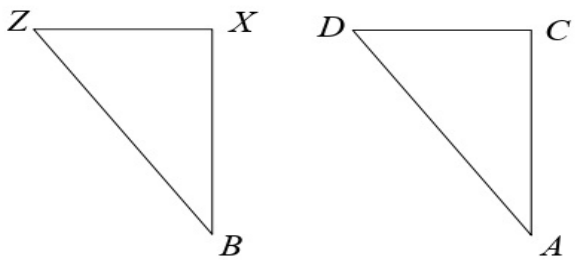
$S\ LQZ \backslash$ rev. cyl.B is less than S inscription.$B \backslash S_{kpð} B$

And this is impossible. Because the surface of the prism is larger than the surface of the cylinder, and the figure inscribed in circle B is smaller than circle B.

Here, I say that with regard to the statement, as shown in the book "Fundamentals", about drawing similar figures: "a figure drawn outside circle A is similar to a figure drawn outside circle B", inside circle A we draw a figure similar to a figure drawn inside the circle IN.

Then, outside circle A we draw a similar figure. This figure will be similar to the figure drawn outside circle B."

As for the statement "the ratio of the outside of circle A of the drawn figure to the outside of circle B of the drawn figure is equal to the ratio of the squares of their radii," it is as follows: the centers A and B of two circles, AC and BX are their radii, CD and ZX let half of the outside circles of the drawn sides be. (Fig. 3)



We carry out AD and BZ. These two triangles are similar. Similar because angles D and Z are equal, like half of equal angles, angles C and X are right angles. The ratio of CD to ZX, that is, the ratio of side to side, is equal to the ratio of the radius to the radius of AC to BX, CD:ZX = AC:BX. The ratio of figure to figure is equal to twice the ratio of side to side, which in turn is equal to the ratio of the square of the radius to the square of the radius.

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The history of Euler identity

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The connection between the sequences of arc values and logarithm values (which had yet to be called functions!) emerged in the 16-th century with the beginning of the study of the isogonal spiral, which was later named the logarithmic spiral. The geometrical and mechanical problems that we are going to discuss also used the logistic curve (which when placed into a coordinate space can be interpreted either as a logarithmic or exponential curve depending on its position) and the hyperbola. As neither the concept of function, let alone the theory of elementary functions existed in the early 18-th century, and the theory of series and integration methods had only just begun developing, in applied tasks geometric methods predominated, building functions, methods of final differences, achieving high virtuosity. Mathematicians and astronomers compared arithmetic and geometric sequences, arcs and segment ratio using a logarithmic scale, which allowed them to obtain results we can now easily obtain using series and definite integrals. Roger Cotes obtained the formula for the surface of an oblate and prolate spheroid containing a verbal expression equivalent to

$$ln(cos\phi + isin\phi) = i\phi,$$

on the basis of which Euler derived this formula 29 years later using integrals, and then series.

Euler identity

$$e^{i\phi} = -1$$

Today this identity receives wide attention from physicists, philosophers and popularizers of science, and ascribed an almost mystical significance: the formula

$$e^{i\phi} + 1 = 0$$

links the five most important numbers in mathematics. Neither Euler himself nor mathematicians of subsequent generations gave particular importance to this formula. Half a century after Euler's actual work, from which the formula follows as a special case, it was first published in an article by the mathematician and engineer Jacques Français, and then another half-century later the astronomer B. Peirce gave a lecture in which he expressed his admiration for its beauty. We will look at all these mathematical events in order.

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Mathematics as the basis for the development of artificial intelligence

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For many centuries, mathematics has been one of the fundamental sciences, used in almost all spheres of human activity. To this day, one can observe the dynamic process of its development, which extends over thousands of years and is still ongoing. It is closely related to the development of society, science, and technology, and it remains a key tool in understanding the world and solving various problems.

Modern mathematics continues to advance on many fronts, including mathematical logic, theory of categories, differential equations, game theory, and many other areas [1]. Mathematics is an integral part of the work of information technology (IT) specialists. Knowledge of mathematics allows IT specialists to develop complex algorithms, analyze data, create effective solutions and predict system behavior. Therefore, professionals in programming need to have good math skills to successfully work in the field of information technology. Mathematics underlies the creation and development of software and the process of solving complex IT problems.

The special role of mathematics can be emphasized in enhancing the professional capabilities of software developers implemented in all spheres of our lives.

First, mathematical knowledge is required to design and analyze algorithms. Algorithms are the basis of software and applications, and their effectiveness is directly dependent on the correct application of mathematical principles. Mastery of mathematics allows IT specialists to create efficient and optimized algorithms, which improve the quality and productivity of the software being developed.

Second, mathematical knowledge plays an important role in the field of artificial intelligence and machine learning. Professionals specializing in these areas must have skills in linear algebra, probability theory, and statistics. Understanding mathematical concepts allows developers to create more accurate and reliable artificial intelligence systems [1-3].

In the modern world, artificial intelligence (AI) plays an increasingly important role in various spheres of life, from automation of production processes to the entertainment industry and medicine. AI is a field of computer science that seeks to create systems and programs capable of performing tasks that previously required human intelligence.

An introduction to the role of mathematics in AI includes an examination of various mathematical concepts and their application to a variety of problems facing researchers and developers in this field. Concepts such as linear algebra, probability theory, optimization, theory of graphs, and others are the main tools used in building AI models and training them based on data.

Mathematics is crucial in various AI aspects, including:

- Data modeling: Utilizing statistical methods, linear algebra, and probability models for data analysis;

- Machine learning and deep learning: Underpinned by mathematical optimization (e.g., gradient descent), probability theory, and linear algebra;

- Information theory: Measuring data information and developing data compression, encoding, and transmission algorithms;

- Optimization methods: Essential for parameter setting and solving uncertainty problems, with techniques like convex optimization;

- Graph theory: Used in AI algorithms to model object relationships and dependencies;

- Problem formalization: Enabling the creation of algorithms and systems to solve computational problems [4].

Thus, mathematics provides the tools and fundamental concepts needed to develop and understand artificial intelligence algorithms, making it indispensable in this field.

In this study, articles containing evidence and experimental bases on the most pressing issues related to the use of data mining methods in the e-learning system were analyzed.

In [2], the use of data mining methods in education for decision-making support was validated. The article introduces software for data analysis using complex classification algorithms. Study [3] explores applying data mining and machine learning tools to small data sets, offering practical results without reviewing popular methods.

It is also necessary to emphasize the significant role of intelligent systems (IS) for educational purposes, which provide unique opportunities for learning, analysis, and personalization of the educational process. There are several ways to use ISs in educational systems: personalized learning, adaptive educational platforms, assessment and feedback, individual progress monitoring, online courses, massive open online courses (MOOC), and interactive educational materials. The use of intelligent systems in education can significantly improve the accessibility, quality, and efficiency of the educational process, making it more adaptive and personalized for each student.

In general, mathematics provides a formal and coherent framework for solving a variety of problems in the field of artificial intelligence. Effective use of mathematics allows us to create more accurate, powerful, and adaptive AI systems that can effectively solve complex problems in the real world.

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MECHANICS

Hydrodynamics of the movement of an oil-water mixture in a pipe

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The movement of oil in the pipeline occurs together with water. This affects the hydrodynamics of the mixture. It should be noted that to date, not enough attention has been paid to the hydrodynamics of the movement of the oil-water mixture in a pipe and remains poorly understood. Therefore, modeling and studying the movement of the oil-water mixture is of great practical and scientific importance.

Let the oil-water mixture move in the pipeline. The equations of continuity and phase motion taking into account mass transfer will have the following form [1-3].

$$\begin{aligned}
&-\tfrac{1}{c_1^2}\tfrac{\partial p}{\partial t}=\rho_1\tfrac{\partial(\varphi_1 v_1)}{\partial x}-D_1\rho_1\tfrac{\partial\varphi_1}{\partial x}\\
&-\tfrac{\partial p}{\partial x}=\rho_1\tfrac{\partial(\varphi_1 v_1)}{\partial t}+2a\rho_1\varphi_1 v_1+k(v_2-v_1)+\rho_1\varphi_1 g+\rho_1 v_{01}\tfrac{\partial(\varphi_1 v_1)}{\partial x}\\
&-\tfrac{1}{c_2^2}\tfrac{\partial p}{\partial t}=\rho_2\tfrac{\partial(\varphi_2 v_2)}{\partial x}-D_2\rho_2\tfrac{\partial\varphi_2}{\partial x}\\
&-\tfrac{\partial p}{\partial x}=\rho_2\tfrac{\partial(\varphi_2 v_2)}{\partial x}+2a\rho_2\varphi_2 v_2+k(v_1-v_2)+\rho_2\varphi_2 g+\rho_2 v_{02}\tfrac{\partial(\varphi_1 v_2)}{\partial x}
\end{aligned}\tag{1}$$

The continuity equations for oil and water can be reduced to the following form

$$\begin{aligned}
&-\tfrac{\partial\rho_{1n}}{\partial t}=\tfrac{\partial(\rho_{1\mathrm{n}}v_1)}{\partial x}-D_1\tfrac{\partial\rho_{1\mathrm{n}}}{\partial x}\\
&-\tfrac{\partial\rho_{2\mathrm{n}}}{\partial t}=\tfrac{\partial(\rho_{2\mathrm{n}}v_2)}{\partial x}-D_2\tfrac{\partial\rho_{2\mathrm{n}}}{\partial x}
\end{aligned}\tag{2}$$

$$\begin{aligned}
&\rho_{1\mathrm{n}}=\rho_1\varphi_1,\quad \rho_{2\mathrm{n}}=\rho_2\varphi_2\\
&\varphi_1+\varphi_2=1
\end{aligned}\tag{3}$$

Having accepted $D_1 = D_2$.

Substituting expression (3) into equation (2) we get

$$\begin{aligned} -\tfrac{\partial \varphi_1}{\partial t} &= \tfrac{\partial(\varphi_1 v_1)}{\partial x} - D\tfrac{\partial \varphi_1}{\partial x} \\ \tfrac{\partial \varphi_1}{\partial t} &= -\tfrac{\partial(\varphi_2 v_2)}{\partial x} + D\tfrac{\partial \varphi_1}{\partial x} \end{aligned} \tag{4}$$

Adding the first and second equations of system (4) term by term we get

$$\frac{\partial v_1}{\partial x} = \frac{\partial v_2}{\partial x} \tag{5}$$

Having differentiated the first equation of system (1) with respect to time t, and the second equation with respect to the x coordinate, subtracting one from the other, we obtain

$$-\frac{1}{c_1^2}\frac{\partial^2 p}{\partial t^2} + \frac{\partial^2 p}{\partial x^2} = \mathrm{D}\rho_1 \frac{\partial^2 \varphi_1}{\partial x \partial t} - 2a\rho_1 \frac{\partial(\varphi_1 v_1)}{\partial x} - \rho_1 g \frac{\partial \varphi_1}{\partial x} - \rho_1 \nu_{01} \frac{\partial^2(\varphi_1 v_1)}{\partial x^2} \tag{6}$$

Following Leibenzon L.S. from the first equation of system (1) we have

$$\frac{\partial(\varphi_1 v_1)}{\partial x} = -\frac{1}{c_1^2}\frac{\partial}{\partial \mathrm{t}} \cdot \frac{1}{\rho_1(1 - \frac{D_1}{v_{10}})} \tag{7}$$

Then from equation (6) taking into account expression (7) and $\frac{D}{\nu_{10}} << 1$ we get

$$\frac{\partial^2 p}{\partial t^2} = c_1^2 \frac{\partial^2 p}{\partial x^2} - (2a + \frac{g}{v_{10}})\frac{\partial p}{\partial t} - v_{10}\frac{\partial^2 p}{\partial x \partial t} \tag{8}$$

Initial and boundary conditions

$$p\big|_{t=0} = p_c(0) - \frac{p_c(0) - p_w(0)}{\ell}x \;\; 0 < x \le \ell, \tag{9}$$

$$\left.\frac{\partial p}{\partial t}\right|_{t=0} = 0 \;\; 0 < x \le \ell\,, \tag{10}$$

$$p\big|_{x=0} = p_c(t) \;\; t > 0, \tag{11}$$

$$p\big|_{x=l} = p_w(t) \;\; t > 0. \tag{12}$$

By solving the differential equation under initial and boundary conditions (9),(10),(11) and (12), we find P. Next, substituting P in place in the second equation of system (1), the mass flow rate of oil is determined.

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Process of instantly discharge of liquid under high pressure in a horizontal straight pipeline

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In the thesis, a horizontal straight pipeline with liquid under high pressure was considered. At a certain moment in time, the outlet of the pipeline is instantly opened and the liquid is discharged. We place the origin of the coordinate axis x at the left end of the pipeline and direct it towards the fluid flow (Fig. 1).

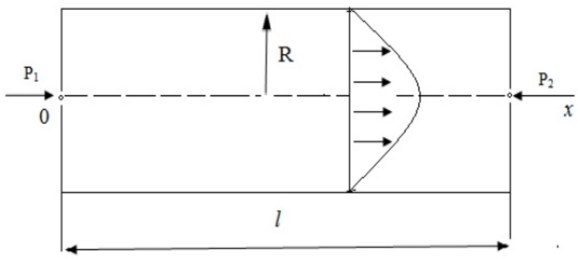


Fig.1. Calculation scheme

The issue was resolved on the basis of material balance. Let's determine the mass m of liquid in the pipeline at any instant:

$$m = \pi R^2 \int_0^l (1 + 2\frac{w}{R})\rho dx \tag{1}$$

where - ρ the density of the liquid, R- the radius of the pipeline, w -the deformation of the pipeline wall.

Boundary and initial conditions

$$P\,|_{x=0} = P_1(t)\;, P\,|_{x=l} = P_2\;(t),\; P_2|_{t=0} = P_0$$

since, for the distribution of the fluid pressure along the axis of the pipeline, which satisfies the interpretation conditions in the first approximation, let us accept the following expression:

$$P(x,t) = P_1(t) - \frac{P_1(t) - P_2(t)}{l^2}x^2 \tag{2}$$

where $P_1(t)$ $P_2(t)$- is the pressure at the left and right ends of the pipeline, respectively (Fig. 1).

Suppose that the density of a liquid varies linearly with pressure:

$$\rho = \rho_{atm}\left[1 + \frac{P(x,t) - P_{atm}}{k}\right] \tag{3}$$

where ρ_{atm} is the density of the liquid at atmospheric pressure, k - the volume of the liquid is the compressibility modulus.

At each moment of time, the mass flow rate of the liquid is

$$G = -\frac{dm}{dt}. \tag{4}$$

Then from expression (4) with formula (1)-(3) we obtain

$$G = -\pi R^2 l \frac{\rho_{atm}}{k} \frac{2\dot{P}_1(t) + \dot{P}_2(t)}{3}. \tag{5}$$

We assume that the mass flow rate of the liquid flowing out of the pipeline after its opening linearly depends on the pressure drop:

$$G = \alpha(P_2(t) - P_{atm}). \tag{6}$$

Here α is the proportionality factor and $Patm$ is the atmospheric pressure.

If we equate expressions (5) and (6), we get the differential equation[3], and considering the initial condition, we get:

$$P_2 = P_0 e^{-\beta t} + P_{atm}(1 - e^{-\beta t}) - 2\int_0^t \dot{P}_1(\tau) e^{-\beta(t-\tau)} d\tau \tag{7}$$

where $\beta = \frac{3\alpha k}{\pi R^2 l \rho_{atm}}$.

From expression (2) with (7) we obtain

$$P = P_1 - \frac{P_1 - P_0 e^{-\beta t} - P_{atm}(1 - e^{-\beta t}) + 2\int_0^t \dot{P}_1(\tau) e^{-\beta(t-\tau)} d\tau}{l^2} x^2 \tag{8}$$

where

$$P_1(t) = P_0 - c_0 t. \tag{9}$$

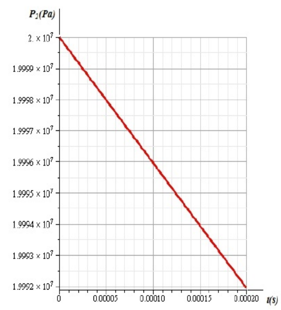
Taking into account the expressions (7) and (9) in the formula (6) of the liquid mass consumption during the sudden opening of the right end of the pipeline, we find:

$$G = \alpha \left[P_0 e^{-\beta t} + P_{atm}(1 - e^{-\beta t}) - 2\frac{c_0}{\beta}(1 - e^{-\beta t}) - P_{atm} \right]. \tag{10}$$

Using the following practical values of the system parameters, we obtain analytical expressions for the pressure change and fluid consumption at the right end of the pipe.

Based on the above values of the system parameters, using formulas (8), (10) numerical calculations were made, graphs were constructed based on the obtained results and given in graphs 1-2, it was shown that the liquid inside the pipeline was emptied in a very short time.

As can be seen in graphs 1-2, the pressure and fluid consumption at the pipeline outlet drop almost immediately.



Graph 1 Pressure dynamics at $\alpha = 10^{-5} \dfrac{kq \cdot m^2}{N \cdot s}$

Graph 2 Dynamics of liquid flow rate at $\alpha = 10^{-5} \dfrac{kq \cdot m^2}{N \cdot s}$

Simulation the dynamics of fluid movement in a pipe taking into account mass transfer

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At high pressures, the movement of a compressible fluid occurs with mass transfer. To the density of the flow of motion is added the transfer of mass from a high density to a lower one, which is determined by the generalized Fick law

$$M = \rho u - D\frac{\partial \rho}{\partial x} \tag{1}$$

Here ρ - is the flux density, D- is the diffusion coefficient, u- is the flow velocity averaged over the cross-section.

The continuity equation in this case has the form:

$$-\frac{\partial \rho}{\partial t} = \frac{\partial M}{\partial x} \tag{2}$$

If $D = \text{const}$ from equation (2) taking into account expression (1) we obtain

$$-\frac{1}{c^2}\frac{\partial p}{\partial t} = \frac{\partial Q}{\partial x} - D\frac{\partial^2 \rho}{\partial x^2} \tag{3}$$

$$Q = \rho u$$

The movement of liquid in a pipe, taking into account the convective term, has the form

$$-\frac{\partial p}{\partial x} = \frac{\partial Q}{\partial t} + 2aQ + u_0\frac{\partial Q}{\partial x} + \rho g. \tag{4}$$

Having differentiated equation (2) with respect to time t, and equation (4) with respect to the x coordinate, we subtract one equation from the other. Further defining $\frac{\partial Q}{\partial x}$ and $\frac{\partial^2 Q}{\partial x^2}$ from equation (3) and substituting them in place we will get

$$\frac{\partial^2 p}{\partial t^2} - \left(c^2 + 2aD\right)\frac{\partial^2 p}{\partial x^2} + 2a\frac{\partial p}{\partial t} + u_0\frac{\partial^2 p}{\partial x \partial t} - Du_0\frac{\partial^3 p}{\partial x^3} - D\frac{\partial^3 p}{\partial x^2 \partial t} = 0 \tag{5}$$

Equation (5) is an equation of fluid motion taking into account mass transfer and the convective term.

Initial and boundary conditions:

$$\left. p\right|_{t=0} = p_c(t) - \frac{p_c(t) - p_w(t)}{l}x, \quad 0 < x \le l \tag{6}$$

$$\frac{\partial p}{\partial t} = 0, \quad t > 0, \tag{7}$$

$$\left. p\right|_{x=0} = p_c(t), \quad t > 0. \tag{8}$$

We will look for a solution to equation (5) satisfying boundary conditions (7) and (8) in the form [3]:

$$p = p_c(t) - \frac{p_c(t) - p_w(t)}{l}x + \sum_{i=1}^{\infty}\varphi_i(t)\sin\frac{i\pi x}{l} \tag{9}$$

Substituting expression (9) into equation (5) and multiplying both sides of the resulting equation by $\sin\frac{i\pi x}{l}$ and integrating it from *0* to l we get the following expression

$$\ddot{\varphi}_i(t) + \left(2a + \frac{Di^2\pi^2}{l^2}\right)\dot{\varphi}_i(t) + \frac{i^2\pi^2\left(c^2 + 2aD\right)}{l^2}\varphi_i(t) =$$

$$= -\frac{2}{i\pi}\ddot{p}_c(t) + \frac{2\cdot(-1)^i}{i\pi}\ddot{p}_w(t) + \dot{p}_c(t)\left(\frac{2u_0}{i\pi l} - \frac{2u_0\cdot(-1)^i}{i\pi l} - \frac{4a}{i\pi}\right) +$$

$$+\dot{p}_w(t)\left(\frac{4a\cdot(-1)^i}{i\pi} - \frac{2u_0}{i\pi l} + \frac{2u_0\cdot(-1)^i}{i\pi l}\right) \tag{10}$$

Applying the Laplace transform from equation (10) we obtain

$$\bar{\varphi}_i = \frac{s\varphi_i(0)}{(s+h)^2 + \omega_i^2} + \frac{\dot{\varphi}_i(0)}{(s+h)^2 + \omega_i^2} +$$

$$+\frac{\left(2a + \frac{Di^2\pi^2}{l^2}\right)\varphi_i(0)}{(s+h)^2 + \omega_i^2} - \frac{2}{i\pi}\frac{\overline{\ddot{p}}_c}{(s+h)^2 + \omega_i^2} +$$

$$+\frac{2\cdot(-1)^i}{i\pi}\frac{\overline{\ddot{p}}_w}{(s+h)^2 + \omega_i^2} + \left(\frac{2u_0\left(1-(-1)^i\right)}{i\pi l} + \frac{4a}{i\pi}\right)\frac{\overline{\dot{p}_c}}{(s+h)^2 + \omega_i^2} +$$

$$+\left(\frac{4a(-1)^i}{i\pi} + \frac{2u_0\left((-1)^i - 1\right)}{i\pi l}\right)\overline{\dot{p}_w} \tag{11}$$

$$\omega_i^2 = \frac{i^2\pi^2\left(c^2 + 2aD\right)}{l^2} - h^2; 2a + \frac{Di^2\pi^2}{l^2} = 2h$$

Let us determine the mass flow rate of liquid per unit area of the pipe flow section. To do this, apply the Laplace transform to the expression (9). Substituting formulas (11) into the resulting expression and then into equation (4), we obtain

$$\bar{Q} = \frac{1}{s+2a}\frac{\bar{p}_c - \bar{p}_w}{l} - \sum_{i=1}^{\infty}\frac{i\pi}{l}\frac{1}{s+2a}\bar{\varphi}_i\cos\frac{i\pi x}{l} + \frac{Q(0,x)}{s+2a} +$$

$$+\frac{V_0}{c^2}\left(\frac{1}{s+2a}\overline{\dot{p}_c}-\frac{1}{s+2a}\left(\dot{\bar{p}}_c-\dot{\bar{p}}_w\right)\frac{x}{l}+\sum_{i=1}^{\infty}\frac{1}{s+2a}\dot{\bar{\varphi}}_i\sin\frac{i\pi x}{l}\right)+$$

$$+\frac{V_0 D}{c^2}\sum_{i=1}^{\infty}\frac{i^2\pi^2}{l^2}\frac{1}{s+2a}\dot{\bar{\varphi}}_i\sin\frac{i\pi x}{l} \tag{12}$$

At the end of the pipeline

$$\bar{Q}\big|_{x=l}=\frac{1}{s+2a}\frac{\bar{p}_c-\bar{p}_w}{l}-\sum_{i=1}^{\infty}\frac{i\pi}{l}\frac{1}{s+2a}\bar{\varphi}_i\left(-1\right)^i+\frac{Q\left(0,l\right)}{s+2a}+$$

$$+\frac{V_0}{c^2}\left(\frac{1}{s+2a}\overline{\dot{p}_c}-\frac{1}{s+2a}\left(\dot{\bar{p}}_c-\dot{\bar{p}}_w\right)\right) \tag{13}$$

$Q\left(0,l\right)$ is determined from the stationary regime of fluid flow in the pipe.

A systematic approach to monitoring medical errors in healthcare

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Electronicization of medical services. When we say a unified system, it is clear that the stations providing the emergency service must have uninterrupted communication between themselves and with the main management institution. In other words, it is necessary to ensure the creation of an automated system of medical assistance first. For this, a fairly large amount of sub-tasks must be solved. Let's note the main ones among these sub-tasks:

- Conducting proper preliminary analytics to analyze the process and create requirements;
- Preparation of a functional document based on the created requirements (technical conditions);
- Technical development of software database;
- Preparing the design of relevant screens;

- Preparation of the internal and external parts of the registration department (4 screens for the internal part and 4 screens for the external part);
- The final completion of the first phase of the software and the preparation of the temporary login screen;
- Testing the initial phase of the software and recording the results of the software;
- Presentation of the program to the management;
- Presentation of the software in the Emergency Medical Station and production in the first test environment;

Detection of medical errors. As for the detection and analysis of medical errors, only in recent years steps have been taken to eliminate this problem. Thomas W. Nolan proposed a three-pronged strategy:

1) development of a system for detecting medical errors;

2) defining visible and traceable procedures for timely detection of medical errors;

3) defining procedures to reduce the consequences of errors.

Staeger P., Favrat B., Vader J.-P., Cornuz J. note that two approaches are used to detect medical errors: individual and systemic. The authors of also confirm these ideas.

Individual approach error is directly related to the human factor, and as a result, the person who makes the mistake is a criminal. This approach is hardly objective and fair, because in many cases the combination of various situations and the actions of many people (including the patient) becomes an integral cause of an unpleasant outcome. In this approach, our opinion is the same as.

The systemic approach assumes that people are fallible, mistakes are inevitable, and at the same time, there is a system that includes active errors as well as hidden factors and conditions that cause errors to occur. One can mention the works of supporters of the systematic approach . We have adopted a systematic approach in the work.

In order to prevent negative events, an in-depth analysis of such errors should be carried out in a specially created system based on the consequences of the error. This approach is to find out the causes of the incident, as well as determining who is guilty. This approach can be considered more effective, because it provides an opportunity to analyze the deep roots of the error.

So, in the first step, three main strategies were proposed to ensure patient safety:

error reporting by the security management system; error learning and investigation; establishing fair information sharing in hospitals and physicians' outpatient practices.

Evaluation of medical errors. At the same time, when evaluating the doctor's professionalism, other factors that hinder his work should be taken into account. Let's mention some of them: false call, unfounded call, patient leaving the call address and similar events. These cases are factors that negatively affect the psychological balance of the medical team.

A doctor's error, regardless of whether it is big or small, should be investigated and evaluated from both a quantitative and a qualitative point of view. To evaluate his work performance is equivalent to evaluating the functional.

$$A = A\left\{N;;\ N_i;;\ N_t\ \ (i \geq t)\,;;D_i^m;;\ Y_i^m;;\ S_i^m\right\} \tag{1}$$

Here A is the doctor of the emergency brigade;

N the number of working days of the doctor in a month;

N_i is the number of real calls of the doctor during one working day;

$N_t\ \ (i \geq t)$ is the number of direct participation of the doctor in the call without intervention (false calls, refusal of the doctor, cases where the patient is not at the call address, etc.);

D_i^m initial diagnosis made by the doctor; Y_i^m is the provided first aid; S_i^m is decision-making by the doctor.

Then, in general, evaluating the doctor's work performance in one call is equivalent to evaluating the following three:

$$< d_{i,j}^{m_{i,j}};;\ y_{i,j}^{m_{i,j}};;s_{i,j}^{m_{i,j}} > \tag{2}$$

$d_{i,j}^{m_{i,j}}$ The evaluation of the coefficient is carried out according to the following scale:

correct diagnosis; partially correct; superficial, harmless help; the need for additional help without intervention; preventable error; lack of knowledge; fatal error.

$y_{i,j}^{m_{i,j}}$ The evaluation of the coefficient is carried out according to the following scale:

full, proper assistance; partially sufficient; drug waste; partial wrong assistance; wrong help.

$s_{i,j}^{m_{i,j}}$ The evaluation of the coefficient is carried out according to the following scale:

right decision; superficial approach; wrong decision.

Among the linguistic coefficients included in the expression (2) above, $d_{i,j}^{m_{i,j}}$ it is evaluated in the segment [-3,+3]; $y_{i,j}^{m_{i,j}}$ is evaluated in segment $[-2, +2]$; $s_{i,j}^{m_{i,j}}$ and is evaluated in the segment [-1,+1].

For each doctor, let's accept ***i* *working days per month and max call*** ***j*** per day. Then its activity can be imagined in the form of matrix (3) given below:

$$Q_{A_k}^m = \left\{ \begin{array}{cccc} q_{1,1}^{m_{1,1}} & q_{1,2}^{m_{1,2}} & \cdots & q_{1,15}^{m_{1,15}} \\ q_{2,1}^{m_{2,1}} & q_{2,2}^{m_{2,2}} & \cdots & q_{2,15}^{m_{2,15}} \\ q_{3,1}^{m_{3,1}} & q_{3,2}^{m_{3,2}} & \cdots & q_{3,15}^{m_{3,15}} \\ \cdots & \cdots & \cdots & \cdots \\ q_{15,1}^{m_{15,1}} & q_{15,2}^{m_{15,2}} & \cdots & q_{15,15}^{m_{15,15}} \end{array} \right\} \tag{3}$$

Waves in a cylindrical net

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Based on the theory [1], [2], the network equations are presented in the form

$$\frac{\partial}{\partial S_1}\left(\sigma_1 \overline{\tau}_1\right) + \frac{\partial}{\partial S_2}\left(\sigma_2 \overline{\tau}_2\right) = 2\rho \frac{\partial^2 \overline{r}}{\partial t^2} + P\overline{n} \tag{1}$$

$$\frac{\partial}{\partial S_1}\left[\left(1+e_1\right)\overline{\tau}_1\right] + \frac{\partial}{\partial S_2}\left[\left(1+e_2\right)\overline{\tau}_2\right] = 2\frac{\partial^2 \overline{r}}{\partial S_1 \partial S_2} \tag{2}$$

σ_1, σ_2- conditional thread tensions;
S_1, S_2- Lagrangian coordinates of thread particles;
e_1, e_2- relative elongation of threads;
$\overline{\tau}_1, \overline{\tau}_2-$ unit vectors of tangents to the threads;

$\bar{i}-$ unit vector parallel to the cylinder axis;
$\bar{j}-$ unit vector tangent to the cross section of the cylinder;
$\bar{k}-$ unit vector perpendicular to the previous ones
then $\bar{\tau}_1 = cos\gamma_1\bar{i} + sin\gamma_1\bar{j}$; $\bar{t}_2 = cos\gamma_2\bar{i} + sin\gamma_2\bar{j}$
Opening expressions (1) and (2) and accepting symmetry, we obtain

$$\frac{\partial}{\partial S}\left(\frac{s}{(1+e)}\frac{\partial x}{\partial S}\right) = \rho\frac{\partial^2 x}{\partial t^2} \tag{3}$$

Next we show $(1+e)\,sin\gamma = const$ and $\sigma sin\gamma = const$, i.e. $\frac{\sigma}{(1+e)} = const$ and equation (3) is a wave equation, where $a^2 = \frac{\sigma}{1+e}$ represents the speed of wave propagation. As can be seen: although the quantities σ and e are variable, the corresponding ratio is constant.

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Longitudinal - radial vibration of a thick-walled pipe with changing mass taking into account the influence of physical-chemical change of the material

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We consider a longitudinal-radial vibration of a thick-walled polymer pipe located in aggressive liquid medium with regard to change of the mass and physical-chemical properties of the pipe material. For studying problems of mechanics of polymers operating in aggressive medium, in the papers [1, 3] a parameter of the relative change of the mass $\lambda(t)$ of a polymer material over time of the form

$$\lambda(t) = \frac{m\ (t) - m_0}{m_0}$$

was introduced.

By means of this parameter $\lambda(t)$ a generalized Hooke law is proposed with regard to the influence of aggressive liquid medium in the form

$$\sigma_{ij} = a_0 \cdot \psi(\lambda) \cdot \varepsilon \cdot \delta_{ij} + 2G_0 \cdot \varphi(\lambda) \cdot \varepsilon_{ij} - \eta_0 \eta(\lambda) \cdot \alpha \cdot \tilde{\lambda} \cdot \delta_{ij}$$

Here $\lambda(t)$ is a relative change in the mass of the pipe material in time, the functions $\psi(\lambda)$, $\varphi(\lambda)$, $\eta(\lambda)$ are correction factors determined by a special experimental - theoretical method. In this case, vibration equations both in longitudinal and transverse directions, with regard to the changing mass and physical-chemical change of the pipe material in time will be:

$$\frac{d^2w}{dr^2} + \frac{1}{r} \cdot \frac{dw}{dr} - \frac{w}{r^2} =$$

$$= m_0 \cdot \frac{1 + \lambda(t)}{\frac{1}{3} \cdot a_0 \cdot \psi[\lambda(t)] + 2G_0 \cdot \varphi[\lambda(t)]} \cdot \left[\frac{d^2w}{dt^2} + \frac{1}{1+\lambda(t)} \cdot \frac{d\lambda(t)}{d(t)} - \frac{dw}{dt}\right]$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\rho_0 \cdot (1+\lambda)}{\frac{1}{3} \cdot a_0 \cdot \psi(t) + 2G_0 \cdot \varphi(t)} \cdot \left[\frac{d^2u}{dt^2} + \frac{1}{1+\lambda} \cdot \frac{d\lambda(t)}{d(t)} \cdot \frac{du}{dt}\right] \tag{1}$$

The principal peculiarity of the obtained dynamics equations (1) is the following:

- character of appearance of the mechanical form of dynamics arising only from mechanical action is represented in these equations by means of the term $\frac{d^2u}{dt^2}$,

- appearance in both equations of (1) the presence of the term $\frac{1}{1+\lambda} \cdot \frac{d\lambda(t)}{d(t)} \cdot \frac{dw}{dt}$ indicates that in the considered vibration problem a dynamic type effect appears, and this effect appears only from the action of physical-chemical origin, i.e. from physical-chemical change of the pipe material. This is a newly identified physical factor.

- appearance in both equations of (1) the presence of the term $m_0 \cdot 1 + \lambda(t)$ indicates on the type of variability of pipe mass in time. Such a character of mass change will indicate the nature of the longitudinal and transverse vibrations of a polymer pipe in aggressive liquid medium in the following form: an amplitude and natural frequency will slowly slow down, vibration period of the pipe will increase in time depending on the degree of its mass variability and physical-chemical variability of the pipe material.

- scientific results of these studies can be used when designing structures made of polymer and composite materials designed to withstand dynamic conditions in aggressive liquid media.

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New wave function and its new differential equation in quantum physics. The fuzzy logic and fuzzy sets theory of L. Zadeh

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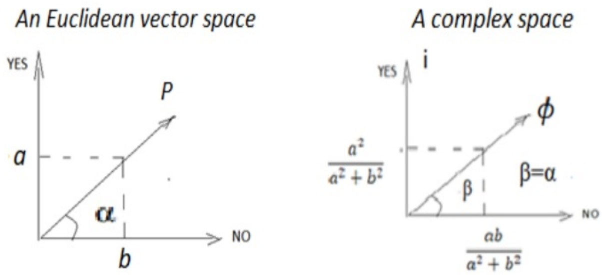
Frequently probability can be presented as the vector P (fig.1), where on axis x and y it is noted "No" and "Yes" events for $N = (a + b)$ objects



$P_+ = \frac{a}{a+b}$ (yes) , $P_- = \frac{b}{a+b}$ (not) and $P_+ + P_- = 1$. Thus $P_+ = \frac{1}{1+tga}$. But there is a new measure $M_+ = cos^2 a = \frac{a^2}{a^2+b^2} = \frac{a}{a-ib} \times \frac{a}{a+ib} = \phi \times \phi^*$ calltd by us a new probability. The new wave function will be $\varphi = \frac{a}{a-ib} = \frac{a^2+iab}{a^2+b^2} = A + iB$ as complex number where $A = \frac{a^2}{a^2+b^2}$ and $B = \frac{ab}{a^2+b^2}$. Here $M_- = \frac{b^2}{a^2+b^2} = sin^2 a$ and $M_+ + M_- = 1$ too as P.

In quantum physics, the wave function is complex number $\psi = Ce^{i\gamma}$ where γ is the phase of the wave is disappeared at multiplication ψ on conjugate $\psi^* = Ce^{-i\gamma}$. The mean of ψ is that it is the probability $P = \psi\psi^* = C^2 = const$. So, our new wave function is $\phi = A + iB = cosa\,(cosa + isina) = cosae^{ia}$ contains itself variable $C = cosa$ where α depends on “where” and “when”, that is $\alpha = \alpha(x, t)$. We can find α, knowing that on axis $x(\alpha = 0$ and $P = 1)$ all objects, figuratively speaking, in the right place at the right time, like a frozen world that is $x \equiv t$. For equality let’s put coefficients p (impulse) and E(energy) in front of x and t that is $px = Et$. In the complex space, the real axis x reflects the relapsed events of the determining world with its classical physics. There is unknowledge of human in it about all influences on a given object and man sees in its randomness and uses the theory of probabilities. It means Einstein’s local realism which isn’t suitable for Bohr’s principal stochasticity of the microworld. In the invisible by us microworld $px \neq Et$ and therefore there is an angle or phase $a = \frac{px-Et}{h} \neq 0$ where h is Planck’s constant. Let’s note from new probability $M_+ = cos^2 a$ appears the L.Zadeh’s measure of possibility $\Pi_+ = cosa$. Because of $cosa > cos^2 a(0 \leq \alpha \leq \frac{\pi}{2})$, $\Pi_+ + \Pi_- > 1$ where $\Pi_- = sina$ is the possibility of opposite event. Such a situation takes place with the Schrodinger’s cat, which alive and dead in same time. Our new probability is $M = \phi \times \phi^* = \Pi e^{ia} \times \Pi e^{-ia} = \Pi^2 = cos^2 a$. Thus we have the new wave function $\phi = \Pi e^{ia} = cosae^{ia} = \frac{1}{2} + \frac{1}{2}e^{2ia} = \frac{1}{2}cos2a + \frac{1}{2}i \sin 2a + \frac{1}{2}$ for which there is a non homogenous differential equation second order $\frac{\partial^2}{\partial a^2}\phi + 4\phi = 2$.

We know $\alpha = \alpha(x,t) = \frac{px-Et}{h}$ and $\Delta = \frac{\partial^2}{\partial x^2} = \frac{\partial}{\partial x}\frac{\partial}{\partial x} = \left(\frac{\partial}{\partial a}\frac{\partial a}{\partial x}\right)^2 = \left(\frac{p}{h}\frac{\partial}{\partial a}\right)^2 = \frac{p^2}{h^2}\frac{\partial^2}{\partial a^2}$, but $\frac{p^2}{h^2} = \frac{m^2v^2}{h^2} = \frac{2mE}{h}$. Therefore $\frac{\partial^2}{\partial a^2} = \frac{h^2}{2mE}\Delta$. So we get the new homogeneous differential equation $\frac{h^2}{2m} \bigtriangleup \phi + 4E\phi = 2E$ with decision $\phi = cos\frac{px-Et}{h}e^{\,i\ \frac{px-Et}{h}} = \frac{1}{2} + \frac{1}{2}e^{2\ i\ \frac{px-Et}{h}}$. It looks like on Schrodinger Equation $\frac{h^2}{2m} \bigtriangleup \psi - E\psi = 0$ for free particle with decision $\psi = Ae^{i\ \frac{px-Et}{h}}$, where $A = const$. If $A = \frac{1}{2}$ and phase of wave will be increased 2 time then $A = \psi + \frac{1}{2}$, that is Zadeh's possibility $\Pi e^{i\alpha} = \phi$ more than wave function ψ in the quantum physics.

For the particle in a potential well $U(0) = U(L) = 0$ and $\phi\,(0) = \phi\,(L) = 0$ our new equation is $\ddot{\phi} + 8k^2 = 4k^2$ where $k^2 = \frac{2m(E-U)}{h^2}$. The decision is $\phi(x) = Ae^{2\sqrt{2}\ \ kxi} + Be^{-2\sqrt{2}\ \ kxi} + \frac{1}{2} = (A+B)\,cos2\sqrt{2}\ kx + i\,(A-B)\,sin2\sqrt{2}\ kx + \frac{1}{2}$ (1). From the initial and boundary conditions $\phi\,(0) = (A+B) + \frac{1}{2} = 0$ we find $(A+B) = -\frac{1}{2}$. Let's find $C' = (A-B)$ from $\phi\,(x) = -\frac{1}{2}cos2\sqrt{2}\ kx + iC'sin2\sqrt{2}\ kx + \frac{1}{2}$ $= sin^2\sqrt{2}\ kx + iC'sin2\sqrt{2}\ kx = sin\sqrt{2}\ kx\left(sin\sqrt{2}\ kx + \right.$ $\left. + iCcos\sqrt{2}\ kx\right)$, where $C = 2C'$. We said above $\phi\,(L) = sin\sqrt{2}\,kL\left(sin\sqrt{2}\ kL + \right.$ $\left. + iCcos\sqrt{2}\ kL\right) = 0$ therefore $sin\sqrt{2}\ kL = 0$ and $k = \frac{pn}{\sqrt{2}L}$. Thus $\phi_n\,(x) = sin^2\sqrt{2}\ \frac{pn}{\sqrt{2}L}x + iCsin2\sqrt{2}\ \frac{pn}{\sqrt{2}L}x = sin^2\ \frac{pn}{L}x + iCsin\frac{2pn}{L}x$. The conjugate $\phi_n^*\,(x) = sin^2\ \frac{pn}{L}x - iCsin\ \frac{2pn}{L}x$. We know the normalization condition

$$\int_0^L \|\phi\,\|^2 dx = \int_0^L \phi_n\,(x)\phi_n^*\,(x)\,dx = \int_0^L (sin^4\ \frac{pn}{L}x + C^2sin^2\ \frac{2pn}{L}x)dx = 1.$$

There are ready formulae

$$\int sin^2ydy = \frac{y}{2} - \frac{1}{4}sin2y$$

and

$$\int sin^4ydy = -\frac{1}{4}sin^3ycosy + \frac{3}{4}\int sin^2ydy = -\frac{1}{4}sin^3ycosy + \frac{3}{4}\,\frac{y}{2} - \frac{3}{16}\ sin2y.$$

If $\frac{2\pi n}{L}x = y$ and $dx = \frac{L}{2\pi n}dy$, then

$$\int_0^L \|\phi\,\|^2\ dx = C^2\ \left(\frac{2\pi n}{2L}L - \frac{1}{4}sin\frac{4\pi n}{2L}L\right)\frac{L}{2\pi n} - sin^3\frac{\pi n}{2L}Lcos\frac{\pi n}{2L}L+$$

$$+\left(\frac{3}{4}\frac{\pi n}{L}L-\frac{3}{16}\sin\frac{4\pi n}{L}L\right)\frac{L}{2\pi n}=C^2\left(\pi n-0\right)\frac{L}{2\pi n}+\left(\frac{3}{4}\pi n-0\right)\frac{L}{2\pi n}=$$
$$=\frac{C^2L}{2}+\frac{3}{8}L=\left(\frac{C^2}{2}+\frac{3}{8}\right)L=1.$$

So, $C^2=\frac{8-3L}{4L}$ and $C^{'}=A-B=\frac{C}{2}=\sqrt{\frac{8-3L}{8L}}$. Thus for equation above (1) we get $\phi=-\frac{1}{2}cos2\sqrt{2}\ kx+i\sqrt{\frac{8-3L}{8L}}sin2\sqrt{2}\ kx+\frac{1}{2}$. From above $k=\frac{pn}{\sqrt{2}L}$ let's find $E_n=\frac{p^2}{2m}=\frac{(hk)^2}{2m}=\frac{p^2h^2}{4mL^2}n^2$ are the quantum of energy too as in quantum mechanics $E_n=\frac{p^2h^2}{2mL^2}n^2$. But unlike the wave function there $\psi_n=\sqrt{\frac{2}{L}}\ i\ sin\frac{\pi n}{L}x$ our wave function is $\psi_n=sin^2\frac{\pi n}{L}x+\sqrt{\frac{8-3L}{8L}}\ i\ sin\frac{2\pi n}{L}x$ which contains real part of complex number that means its join with reality. In the quantum mechanics the probability is $P=\int_0^L\phi_n\phi_n^*=\frac{2}{L}sin^2\frac{\pi n}{L}x$ but our new probability is $M=\int_0^L\phi_n\phi_n^*=sin^4\frac{\pi n}{L}x+\frac{8-3L}{8L}\ sin^2\frac{2\pi n}{L}x=(sin^4\frac{\pi n}{L}x-\frac{3}{8}sin^2\frac{\pi n}{L}x)+\frac{1}{L}sin^2\frac{2\pi n}{L}x.$

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Research and modeling of helicopter main rotor blade. Rectangular and curve blade shapes

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The aerodynamics of the wing or blade element is almost the main part in the helicopter aerodynamics. It is obvios that the top view shape of these propeller blades is usually rectangular.

As a result of the collision of rectangular blade with the air, a great noise is generated, when the speed of the tip section of the blade reaches 180-200 m/s, the air flow is interrupted, thus vibration occurs and the drag increases, therefore, the lift decreases, etc. In the end, the aerodynamic quality of such propellers decreases.

Aerodynamics changes according to the shape of the top view of curve shaped blade, the contacting of nose parts of the blade with air molecules has not been at the same time, but they collide at different time segments. Therefore, noise, vibration, form drag, oscillating movements are reduced.

The aerodynamic indications of straight flow - rectangular and curve shaped blades at the same number of cycles and same radius were investigated and the results were presented.

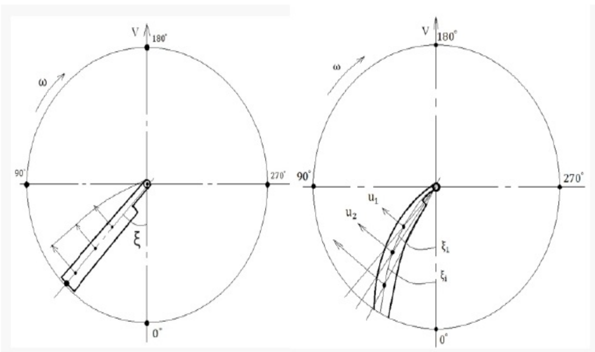

Fig 1. Azimuth angles of Rectangular and Curve shaped blades at sections

The nature of distribution of air forces along a rectangular airfoil is already known to science, and in particular airfoils, the center of pressure (CP) locates on the $(0.70 \div 0.75)$ R section of the radius. The nature of the distribution of air forces on the curve shaped blades is unknown.

Let's conduct research to compare the aerodynamic parameters of a curve shaped blade and rectangular blade.

The azimuth angle is the same in the cross sections of rectangular blades, but it is variable in the cross-sections of curve shaped blades. The linear velocity is found by the expression U $= \omega R$, where $\omega = 2\pi n$ is the angular speed and n is the number of revolutions. Each blades of both propellers creates lift $Y = c_y \frac{\rho v^2}{2} S$. In this formula $q = \frac{\rho v^2}{2}$ dynamic pressure and area S are the same. The lift coefficients (C_L) of rectangular and curve shaped blades are different, which indicates the nature of distribution of air forces along the radius is different.

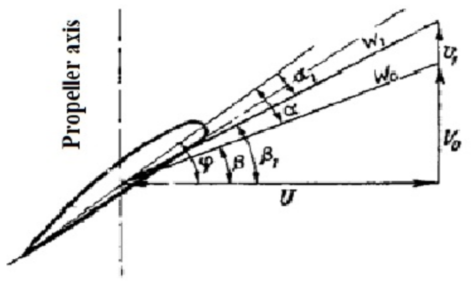

Fig 2. The velocity triangles of both cross sections

The velocity triangles of both airfoil cross sections will be as shown in Fig.1. The lift of the airfoil cross section

$$dY = c_y \rho b dr \frac{W^2}{2} \tag{1}$$

The drag of the airfoil cross section

$$dX = c_x \rho b dr \frac{W^2}{2} \tag{2}$$

It is known from aerodynamics that the form drag (or parasite drag) of the nose and the lift coefficient (C_L) depend on $\overline{r} = \frac{r}{R}$ the relative radius. From the theory of inductive resistance, such blades can not create an inductive speed, in other words unable to create lift. Thus, if we consider the airfoil cross section as an infinitely extendable wing, then the induced velocity generated by the propeller must be found by some other method.

Let's project the Lift and Drag of the cross section in two directions - parallel to the axis of the propeller and perpendicular to it. Then, we get the drag of the blade element

$$dP = dY cos\beta - dX sin\beta \tag{3}$$

If we replace the received expressions (1) and (2) in the expression (3), we can get the drag of the section.

$$dP = (c_y cos\beta - c_x sin\beta)\, \rho b dr \frac{W^2}{2} \tag{4}$$

The thrust (lift and drag) depends on the distribution of air forces along the radius.

In order to obtain the nature of the distribution of air forces along the radius, the analysis of propellers with rectangular and curve shaped blades was carried out by using Ansys program. There is a large disruption of the air flow on the rectangular airfoil, and the center of pressure (CP) of the airfoil is at 75% of the radius, while on the curve shaped airfoils, the airflow disruption is reduced and the center of pressure locates at 95% of the radius.

Therefore, the aerodynamic quality of curve shaped propellers is greater than that of rectangular blades.

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Displacement's time of fluid from the annular space into the central channel

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Determining the beginning of the process of displacing the accumulated fluid in the annular space into the central channel is important. Since, over time, the influx of liquid into the well due to an increase in its level in the central channel gradually decreases. There is a certain level of accumulation that delivers maximum oil production [1-7].

The movement of liquid columns without taking into account its compressibility in the annular space and the central channel is described by the equations (Fig. 1)

$$\rho_j(l_3 - x)f_k\ddot{x} = Pf_k - P_cf_k + \rho_j g(l_3 - x)f_k$$

$$\rho_j(l_3 + \frac{f_k}{f_m}x)f_m\frac{f_k}{f_m}\ddot{x} = P_cf_m \tag{1}$$

where l_3- height of the accumulated liquid column in the wellbore, x-coordinate, P_c- bottom hole pressure, f_m-suqare, f_k- cross-sectional area of the annular section, P- interface pressure, g- acceleration of gravity.

Assuming, as a first approximation, the increase in P_{CA} wellhead pressure, which is necessary for forcing fluid from the annular space into the central channel, depending on t time, is linear, we obtain

$$P = k\frac{t}{T_2} + \frac{k\rho_{atm}g}{P_{atm}}l_3(\frac{l}{l_3} - 1 + \xi) \tag{2}$$

where T_2 is the time period during which the liquid in the annular space is completely displaced into the central channel, k- proportionality factor, ρ_{atm} - gas density at atmospheric pressure, l- pipe string running depth, P_{atm} - Atmosphere pressure $\xi = \frac{x}{l_3}$.

From expressions (1) and (2) we obtain

$$\left[1-\xi\left(1-\frac{f_k}{f_T}\right)\right]\ddot{\xi}=\frac{k\frac{t}{T}+\frac{k\rho_{atm}g}{P_{atm}}l_3(\frac{l}{l_3}-1+\xi)}{l_3^2\rho_j(1+\frac{f_T}{f_k})}-\frac{g}{l_3}\xi \tag{3}$$

As a first approximation, to simplify the solution of the problem, we accept $\frac{f_k}{f_T}=1$.

Then from expression (3) we obtain

$$\ddot{\xi}+\frac{g}{l_3}\left(1-\frac{k\rho_{atm}}{2P_{atm}\rho_j}\right)\xi=\frac{k\frac{t}{T}}{2l_3^2\rho_j}+\frac{kg\left(\frac{l}{l_3}-1\right)\rho_{atm}}{2P_{atm}\rho_j l_3} \tag{4}$$

Applying the Laplace transform and the inversion and convolution theorems from expression (4) we obtain

$$\xi=\frac{k}{2l_3^2\rho_j\omega T}\cdot\left(\frac{t}{\omega}-\frac{1}{\omega^2}\sin\omega t\right)+\frac{kg\left(\frac{l}{l_3}-1\right)\rho_{atm}}{2P_{atm}\rho_j l_3}\frac{1}{\omega^2}\left(1-\cos\omega t\right)$$

Or taking into account $x=\xi l_3$ we get

$$x=\frac{k}{2l_3\rho_j\omega T_2}\cdot\left(\frac{t}{\omega}-\frac{1}{\omega^2}\sin\omega t\right)+\frac{kg\left(\frac{l}{l_3}-1\right)\rho_{atm}}{2P_{atm}\rho_j}\frac{1}{\omega^2}\left(1-\cos\omega t\right) \tag{5}$$

where $\omega^2=\frac{g}{l_3}\left(1-\frac{k\rho_{atm}}{2\rho_6 P_{atm}}\right)$

Complete repression will occur when

$$x|_{t=T_2}=l_3 \tag{6}$$

From expression (5) taking into account condition (6) we obtain

$$l_3=\frac{k}{2l_3\rho_j\omega T_2}\cdot\left(\frac{T_2}{\omega}-\frac{1}{\omega^2}\sin\omega T_2\right)+\frac{kg\left(\frac{l}{l_3}-1\right)\rho_{atm}}{2P_{atm}\rho_j}\frac{1}{\omega^2}\left(1-\cos\omega T_2\right)$$

From expression (7) T_2 is determined.

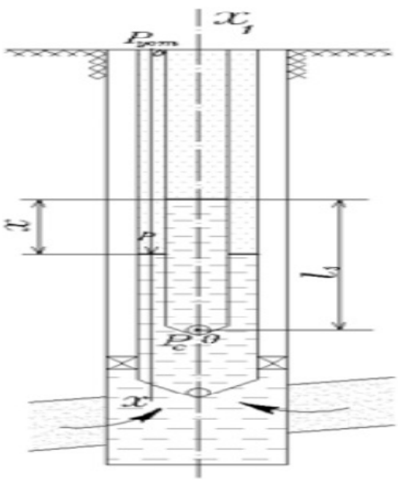

Fig.1.

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On a universal function of corrosion strength of materials with the influence of mechanical stresses and the concentration diffusion of the diffusing substances

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Based on the analysis of the forms of curvatures of dependence of time to corrosion failure (t_0)on the concentration of diffusing substances (C) we offer the following analytic formula for this dependence:

$$t_0 = t_0\left(\sigma,\, T\right) = f\left(\sigma\right) e^{-\alpha\left(\frac{c}{c_s}-1\right)} \tag{1}$$

Here $f\left(\sigma\right)$ is a function experimentally defined from the mechanical stress σ:

C_s is some conventional concentration reduced to pure value selected from the interval of change of concentration C;

α is an experimentally determined constant: $\alpha > 0$:

The function $f\left(\sigma\right)$ can be represented by various formulas. According to (1) it corresponds to the time to corrosion failure under the stress σ for $C = C_s$; $f\left(\sigma\right) = t_0\left(\sigma,\, T_s\right)$. Analysis of experimental dependence $t \sim \sigma$ at a constant concentration C_s shows that the following formula is more suitable for the function $B\left(\sigma\right)$:

$$f\left(\sigma\right) = t_{0s}\left(\frac{\sigma}{\sigma_s}\right)^{-\beta}, \tag{2}$$

where σ_s is stress reduced to pure value that can be selected from the interval of change of the stress σ;

$0 < \beta$ is a constant to be defined;

$t_{0s} - const$ is the time to corrosion failure at $\sigma = \sigma_s$, $C = C_s$.

Taking into account (2), formula (1) takes the form:

$$t_0 = t_0\left(\sigma,\, T\right) = t_{0s}\left(\frac{\sigma}{\sigma_s}\right)^{-\beta} e^{-\alpha\left(\frac{c}{c_s}-1\right)}. \tag{3}$$

Let us formulate a system of experiments [3] that allow to determine universal constants for each system "material-aggressive medium" and that are contained in formula (3). We accept the conjecture about the universality of the function $t_0(\sigma, T)$ for each system "metal-aggressive medium". The accepted conjecture allows to determine unknown constants from various experiments on corrosion failure, for example, from the experiments on corrosion failure under tension or bending of experimental samples in aggressive medium with various constant concentrations of diffusing substances.

Let the quantity C_s be chosen from the change interval of C, the quantity σ_s from the change interval of σ. Stretching the sample under tension σ_s to failure in the aggressive medium of concentration C_s, we find the time t_{0s}. From formula (3), for $C = C_s$ and various $\sigma = \sigma_k = const$ $(k = 1, 2, ..., n)$ we have

$$\beta = \ln\left(t_{0s}/t_0\left(\sigma_k,\ T_s\right)\right)/\ln\left(\sigma_k/\sigma_s\right)\ . \tag{4}$$

Experiments on corrosion failure of samples when stretched by constant stresses σ_k in the aggressive medium at constant temperature and concentration C enables to determine the constant β according to the formula (4). Since $k > 1$, then for determining β it is necessary to use one of the methods of mathematical approximation.

We now determine the constant α. From formula (3) for $\sigma = \sigma_s = const$ and for various constant concentrations $C = C_k = const$ $(k = 1, 2, ..., m)$ we have

$$\alpha = \left[\ln \frac{t_{0s}}{t_0\left(\sigma_s,\, C_k\right)}\right] \Bigg/ \left(\frac{C_k}{C_s} - 1\right)\ . \tag{5}$$

The experiments on corrosion failure of samples when stretched by constant stress σ_s in the aggressive medium with various constant concentrations C_k allow to determine the constant α according to formula (5). When determining α we should also use one of the methods of mathematical approximation.

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Torsional scattered destruction of an anisotropic hollow shaft

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The process of torsional scattered destruction of a cylindrical hollow shaft with variable shear modulus along the radius was studied. The destruction front movement equation was derived and solved. The front extension curves were constructed on the base of the results of numerical calculations.

Delayed strength of hollow shafts, thickwalled cylindrical pipes under statical loading with a torque is determined by temporary destruction process dependent in particular on the size of the torque, rheology, anisotropy of mechanical features of the shaft, its geometry. Such a similar static fatigue phenomenon is explained by the availability of irreversible kinetic process of damage accumulation in the body, by the scattered destruction. Knowledge of the period of the latent and apparent destruction allows sufficient use of the life cycle of the construction.

In the present paper, based on the Suvorova – Akhundov hereditary type damage theory we study the process of scattered destruction of a hollow circular cylinder by the torsional moments applied to its endwalls. The material of the hollow shaft possesses cylindrical axisymmetric anisotropy under which mechanical module depend only on the current radius.

In model relations the damage accumulation process is described by a hereditary type damage operator that in the absence of unloading behaves as an ordinary viscoelasticity operator that allows to use the existing solution methods.

The integral equation of the destruction front motion is a second kind nonlinear Volterra integral equation. In the general case, for an arbitrary form of the kernel of the integral member, its integration is a difficult mathematical problem. However, for some parts of the kernel of the integral term, it is succeeded to obtain solutions in a visible analytic form. This allows to analyze qualitatively the destruction process picture and reveal its most significant features.

A problem of initiation and development of torsional destruction zone in a cylindrical anisotropic hollow cylinder when shear modulus changes along the section. A significant influence of variability of shear modulus on the destruction front propagation velocity is revealed.

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Importance of monitoring of patients after CO poisoning

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Carbon monoxide poisoning is one of the most common types of poisoning and the leading cause of death by poisoning worldwide. Along with the diagnosis of poisoning by carbon monoxide poisoning, the forecast has great importance for the consequences of monitoring. Besides symptoms and signs of central nervous system dysfunction, myocardial damage should also always be considered in the context of CO poisoning. The lack of objective data highlights the need for new randomized studies.

Indoor air pollution sources from outdoor diffusion and indoor human activities have become the leading cause of the disease burden. Numerous poisons, from natural toxins to synthetic chemicals existing in our environment, can produce a wide variety of deleterious effects on living organisms.

Along with the diagnosis of poisoning by carbon monoxide poisoning in order to forecast has great importance for the consequences of monitoring. As, carbon monoxide poisoning can cause damage to tissues and organs such as the cardiovascular and respiratory systems, muscles, liver, and kidneys. Major causes of tissue and organ damage are not only hypoxia, but also oxidative stress, formation of oxygen-reactive species, neuron necrosis, apoptosis, and abnormal inflammation.

Carbon monoxide poisoning, which causes hypoxic insults to the brain and other organs, is a leading cause of mortality and morbidity. Neurological symptoms of CO poisoning can manifest not only immediately but also as late as 2 to 6 weeks after successful initial resuscitation as delayed neurological sequelae (DNS).

Previous studies on conventional MRI have shown that particularly vulnerable areas of the brain include the cerebral cortex, hippocampus, basal ganglia, and cerebellum and that lesions of the globus pallidus are typically seen in the chronic phase of CO poisoning.

I should also note that of patients with CO poisoning treated with normobaric oxygen, nearly half develop cognitive sequelae after 6 weeks. Some possibly permanent sequelae include gait and motor disturbances, peripheral neuropathy, hearing loss and vestibular abnormalities, dementia, psychosis, amnestic syndromes, and Parkinsonism.

The heart is extremely susceptible to CO-induced hypoxia, due to its high oxygen demand. Cardiac involvement manifests mainly as ischemic insult, with elevated enzyme levels and ECG changes ranging from ST-segment depression to transmural infarction. Conduction abnormalities, atrial fibrillation, prolonged QT interval and ventricular arrhythmia have been demonstrated.

Monitoring is advisable for both once poisoned and also for persons affected by chronic intoxication.

Monitoring needs to be conducted after successful treatment in the hospital. Therefore, the starting time of monitoring should coincide with the end of the treatment. Functional parameters and biochemical analysis of carbon monoxide victims need to be examined from time to time during the monitoring (fixed time interval).

Monitoring information is based on diagnostic parameters and the next laboratory, functional, etc. indicators of analyses of finished current treatment parameters. The result of parameter monitoring is a body of measured values of parameters obtained on continuously adjacent to one another time intervals during which the values of the parameters do not change appreciably.

Monitoring performs several organizational functions:

1. it reveals the state of critical or being in the process of change conditions in the status of a patient for whom a plan of future measures will be worked out;

2. it provides data on the previous state giving feedback that will be worked out; relating to earlier successes and failures of a definite policy or programs;

3. it checks on the conformity with regulations and contractual obligations;

Periods may vary in the range of a week, month, quarter, six months, year. To organize monitoring we shall add a module to the intelligent system for differential diagnosis (see Fig. 1).

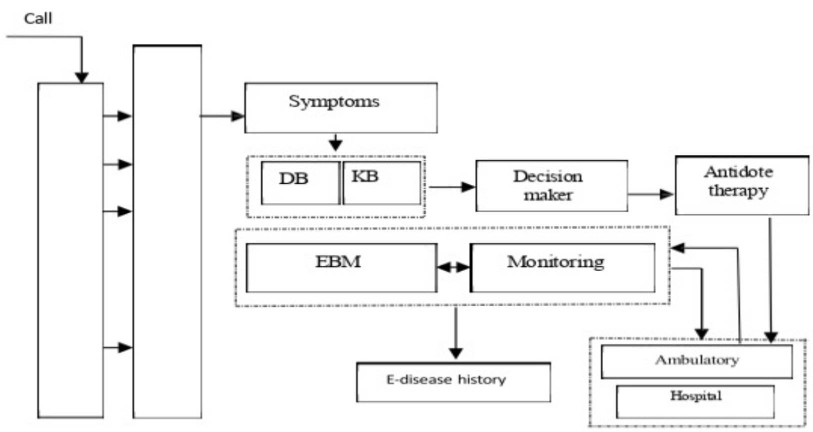

Fig.1. Structure of the intelligent information system

Manna-Whitney's U-criterion, Wilcoxon's T-criterion, Friedman's criterion, and Kraskal-Wallis's H-criterion, which are biostatistical parametric and non-parametric criteria, are used to observe the dynamics of indicators in the time interval. Since the application of the time series method during monitoring reveals whether the change interval of any indicator is within the norm, it is possible to minimize the number of inspections.

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On determining the fatigue loading cycle of the intermediate shaft of a ship shafting

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A problem of determining the number of fatigue cycles of failure of a structural element of ship shafting, intermediate shaft. A ship shifting transmits energy from the crankshaft of the engine to the screw. The shaft material is carbon or alloy steel state. When working, the intermediate shaft is subject to spatial stress. In strength theory, taking into account the spatial stress state is often carried out by introducing equivalent stress.

In conformity to the intermediate shaft, the equivalent stress is determined by the following formula [1]:

$$\sigma_e = \sqrt{\sigma_0^2 + 3\tau_{t_r}^2}, \tag{1}$$

where σ_0 is the greatest normal stress,$\tau_{t_r}$is a shear stress from torsional deformation. In this case, the stress σ_0, is represented as the sum of the compression stress under the action of the stop of the screw (σ_{com}) the greatest stress at bending(σ_{bn}) and stress from the inaccuracy of the shaft line moment (σ_{ins})

$$\sigma_0 = \sigma_{com} + \sigma_{bn} + \sigma_{ins} \tag{2}$$

For the quantities τ_{t_r}, σ_{com},σ_{bn} we have the following expressions [1] :

$$\tau_{t_r} = \frac{20800N_e}{\pi^2 \omega d^3}; \sigma_{com} = \frac{4N_e}{\pi d^2 V}; \tag{3}$$

$$\sigma_{bn} = \frac{32}{\pi d^3}\left(\frac{(R-Q)^2}{2q} + Q_a\right) \tag{4}$$

In formulas (1)-(4) the stresses are expressed by kPa. The following denotations are accepted: N_e is the power transmitted by the shaft (kWt), ω is the shaft rotation frequency $1/s$, d is the diameter of the intermediate shaft (m).

V is the vessel speed (m/s), $\eta = 0,6 \div 0,72$ is the efficiency of the line from the cranked shaft of the engine to the screw depending on the type of transmission; Q is a concentrated load (kN), a is the distance from the support to the application point Q, R is a reaction in the support (kN); q is distributed load from the self-weight of the shaft (kN/m). The quantities R and $q$are determined by the formulas :

$$R = \frac{ql}{2} + Q\frac{l-a}{l}, q = \frac{\pi d^2}{4}\gamma, \tag{5}$$

where l is the length of the intermediate shaft (m), γ is a specific gravity of the shaft material.

Substituting the values (2)-(5) for the quantities τ_{t_r}, σ_{com},σ_{bn} in formula (1), we find a specific expression for the equivalent stress σ_l. In this case, we take into account that in calculations for σ_{ins} we take 30000 kPa.

The problem is in finding the number of loading cycles to the fatigue failure of the intermediate shaft subjected to the stress (1) taking into account (2)-(5). To this end, the Paligrin-Miner formula is used [2]:

$$\int_0^{N_*} \frac{dk}{N_0\left(\sigma_e\left(k\right)\right)} = 1. \tag{6}$$

Here $N_0 = N_0\left(\sigma_e\right)$ is a universal material function of the intermediate shaft. It is determined experimentally as the number of cycles in basic tests for fatigue failure with a constant stress amplitude; N_* is the number of cycles to be determined to fatigue failure under cyclic loading with an arbitrary stress amplitude σ_e. In conformity to our problem, formula (6) was used in the form:

$$\int_0^{\omega t_*} \frac{d\tau}{N_0\left(\sigma_e\left(\omega\tau\right)\right)} = \frac{1}{\omega}, \tag{7}$$

where t_* is a time to fatigue failure.

The material function $N_0\left(\sigma_e\right)$ was approximated in the form:

$$N_0 = N_s \exp\left[2\left(1 - \frac{\sigma_e}{\sigma_s}\right)\right]. \tag{8}$$

Here N_s is an experimentally determined number of cycles to failure for $\sigma_e = \sigma_s$. The quantity σ_s is fixed from the change interval σ_e;$d = const$.

Using the above-determined expression for σ_e in (8), we find the dependence $N_0\left(\sigma_e\left(\omega\tau\right)\right)$ after which the integrand expression in (7) became known. Calculating the integral (7) made it possible to determine the time t_* to fatigue failure of the shaft under consideration. We find the number of cycles to fatigue failure N_* by the formula: $N_* = \omega t_*$.

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On periodic oscillations in the in-situ rheogaschemical reaction of gas formation

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An innovative method of enhanced oil recovery based on a new mechanism of in-situ stoichiometric chemical reaction of carbon dioxide generation is well-known [1]. Chemical reaction mechanism can be described by a system of differential equations expressing the rate of change in the concentrations of the reaction components. Integration of these equations gives the time dependences of the component concentrations, i.e., indicates the temporal development of the chemical reaction.

Concentrations of initial and final products of the gas generation reaction are not subject to fluctuations, and their periodic change is observed only for intermediates [2]. Periodic behavior, not depending on the initial conditions, can be described by differential equations of nonlinear time dependence of the corresponding system variables. In addition, the process of adsorption of reaction products at some stages of gas generation is the cause of the oscillation process.

Under reservoir conditions, the reaction scheme of the model consists of the following stages:

$$A + Y \to X + P$$

$$X + Y \to 2P$$

$A + X \to 2X + 2Z$ - the product of adsorption.

$$2X + Y \to 2Z + P$$

$$2Z + P \to hY, \tag{1}$$

where A is taken as soda ash Na_2CO_3, Y - acid (HCl). Thus, P is the stoichiometric reaction product (acidic medium) of CO_2 gas; X is the reaction product of Na_2CO_3 in NaCl sodium chloride solution, exhibiting spatial and temporal dependence, while A and Y are time-independent as they enter and leave the reaction medium at a constant rate. To describe fluctuations in a chemical system, you must have at least two variables and if, for example, the concentration of X tends to increase, it will continue to increase until it reacts with Y. Y may eventually decrease the concentration of X while its own yield will increase. On the other hand, if too much Y favours the formation of X, an oscillatory change in the concentrations of the reactants will be seen in the reaction [3].

In the limiting case, it is possible to create strongly non-equilibrium reaction conditions by complete suppression of reverse reactions. The advantage of such a technique is that it greatly simplifies the reaction rate equations [4]. These equations can be simplified without affecting the property of the model if we take all rate constants equal to unity and thus introduce dimensionless variables. Taking into account these simplifications, the differential equations for the rheogaschemical reaction in the in-situ gas generation will take the form:

$$\frac{dX}{dt} = A - (Y+1)\,X + X^2 Z; \quad \frac{dZ}{dt} = YX - X^2 Z \tag{2}$$

The solution of the system of equations (2) under the boundary conditions given by the initial values of X and Z completely determines the values of concentrations that can be achieved in the formation reaction zone. With appropriate adjustment of the control parameters, the system can be maintained in a stationary nonequilibrium state in which the rates of change of intermediate substances are zero: $\partial X/\partial t = 0$ and $\partial Z/\partial t = 0$. When these conditions are imposed, equations (2) admit a single homogeneous stationary solution X^s, Z^s.

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Effect of the features of the Earth's crust on wave processes during earthquake

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Usually, during earthquakes, there are two consecutive impulses. The first corresponds to the type of longitudinal waves and appears in the form of oscillatory movements in a plane parallel to the surface of the earth with increasing amplitude. The second is a single instantaneous impulse, most likely corresponding to the surface Rayleigh waves, in which the movement is vertical, i.e. perpendicular to the surface of the earth. We observed that in the city of Lenkoran in the south of the Republic of Azerbaijan during earthquakes the first type of waves is not felt at all even for high-magnitude earthquakes which happened several years ago in this area. In an attempt to find the reason, we turned to the fact of the peculiarities of the structure of the earth's crust in this area. And it differs in that there are a lot of water wells here, and the level of groundwater is quite close to the surface. This level ranges from 3 to 5 meters above the surface of the earth. Taking into account these features and the fact that Lenkoran is located between the Caspian Sea and the Talysh mountains and that usually the epicenters of shocks are located on the seabed, the problem was posed of the non-stationary dynamics of an elastic semi-infinite layer, the lower part of which borders on a compressible ideal fluid (Fig. 1). The fluid movement is considered potential, i.e. irrotational. Some or all of the end face of the layer is subjected to impact.

To solve this problem, we used some results in [1], devoted to the study of the dynamics of rectangular prisms, from the standpoint of the exact three-dimensional theory of elastodynamics. However, in the present problem, the existence of media of different types bordering each other significantly complicates the solution process. We propose a new method for determining the originals of functions — transformations, which in the present work have a very complex form; they are represented through determinants of the fifth rank. In a sense, this method is a generalization of a similar method first proposed in [1], and later for axially symmetric cases in [2]. We developed an exact solution valid in the initial short time of the process, which also provides a good picture of the whole process for subsequent times. The results confirm with high accuracy our hypothesis outlined above.

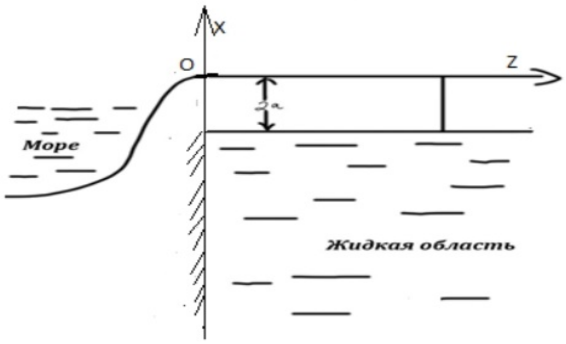

Fig.1.

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Vibrations of a fluid-filled inhomogeneous spherical shell stiffened with inhomogeneous ribs in soil

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It is important to make a construction analysis taking into account the real properties of the material. In the submitted thesis we study vibrations of a fluid-filled inhomogeneous spherical shell stiffened with inhomogeneous ribs in soil.

For solving the problem we use Hamilton – Ostrogradsky variational principle. According to this principle, for total energy of a cylindrical shell is dynamically interacting with fluid and stiffened with inhomogeneous, orthotropic rings in soil in meridional and equator directions as follows:

$$J = U + H + A_q + A_m \tag{1}$$

To take into account inhomogeneity of the shell and rings, the Young modulus and material density were accepted in the form of the coordinate function: $E = E(\varphi, \psi), \rho = \rho(\varphi, \psi)$. For describing the motion of the medium filling the inner domain of the shell, the Lame system of equations was used:

$$a_l^2 graddiv\overrightarrow{u} - a_t^2 rotrot\ \overrightarrow{u} + \rho_s \frac{\partial^2 \overrightarrow{u}}{\partial t^2} = 0. \tag{2}$$

Here a_l, a_t is speed of propagation of longitudinal and lateral waves in soil, ρ_s is the density of the material of the cylindrical shell, $\overrightarrow{u}(s_x, s_\theta, s_r)$ is a displacement vector. The fluid is accepted to be ideal and the motion of the fluid is determined by the equation written in the potential:

$$\nabla^2 \Phi - \frac{1}{a^2} \frac{\partial^2 \Phi}{\partial t^2} = 0. \tag{3}$$

Contact conditions are also added to the total energy of the system (1). Displacements between soil and cover

$$s_r = w,\ s_\varphi = u, s_\psi = \upsilon \quad (r = R) \tag{4}$$

and conditions of pressure equality are written:

$$q_r = -\sigma_{rr}, \quad q_\varphi = -\sigma_{r\varphi}, \quad q_\psi = -\sigma_{r\psi}, \quad (r = R). \tag{5}$$

The frequency equation was established from the stationarity condition of expression (1) and was studied by the numerical method.

Studying the movement of circular embedding in a linear viscous-elastic medium

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The exact solution of problems of movement of a body in viscous and also in viscous-elastic medium is related to great mathematical difficulties. In the paper, the vibration of a solid body in compressible viscous liquid is studied at small values of the Reynolds number. This time the movement of the liquid is described by means of the Navier-Stocks equations taking into account the inertia terms. The solution of the equation is expressed by the scalar and vector potentials.

In the paper, we study nonstationary movement of a circular embedding in a linear viscous-elastic medium. In this case, the law of motion commutation is given beforehand. In the paper, the reaction, reaction moment, and longitudinal reaction of the sphere and medium to the embedding are determined. Since the speeds of these reactions are complex, their asymptotic expansion in the form of the Macleron series is used. The solution of dynamical problem for a viscous-elastic material is obtained from the solution of appropriate problems for an elastic medium by applying the Laplace integral transform to these solutions, replacing the elastic constants by appropriate viscous-elastic complex modules and making inverse transformation and applying the compatibility principle.

In this case, by choosing appropriate constants we take various models. As a result, reaction force, reaction moment, and longitudinal reaction are determined. We can note that using the obtained results, one can find expressions of these reactions for viscous-elastic compressible liquids.

In this work, we study nonstationary movement of circular embedding in a viscous-elastic medium.

The movement of a solid in a linear viscous-elastic medium is characterized by the following relations:

$$a_0 S_{ij} + a_1 \frac{dS_{ij}}{dt} + \cdots + a_m \frac{d^m S_{ij}}{dt^m} = b_0 e_{ij} + b_1 \frac{de_{ij}}{dt} + \cdots + b_n \frac{d^n e_{ij}}{dt^n} \qquad (1)$$

$$C_0 \sigma_{kk} + C_1 \frac{d\sigma_{kk}}{dt} + \cdots + C_l \frac{d^l \sigma_{kk}}{dt^l} = d_0 \varepsilon_{kk} + d_1 \frac{d_1 \varepsilon_{kk}}{dt} + \cdots + d_q \frac{d^q \varepsilon_{kk}}{dt^q} \qquad (2)$$

Here e_{ij} and S_{ij} are the components of deviatoric deformation and deviatoric stress, respectively. These components are related to ε_{ij} and σ_{ij} stress components in the following form:

$$e_{ij} = \varepsilon_{ij} - \frac{1}{3}\delta_{ij}\varepsilon_{kk}, \quad S_{ij} = \sigma_{ij} - \frac{1}{3}\delta_{ij}\sigma_{kk} \qquad (3)$$

here

$$\sigma_{kk} = \sigma_{11} + \sigma_{22} + \sigma_{33} = 3\sigma$$
$$\varepsilon_{kk} = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} = 0$$

$$\delta_{ij} = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases} \qquad (4)$$

Applying the Laplace integral transform

$$\overline{f}\left(\omega\right) = \int\limits_0^\infty \left(t\right) e^{-\omega t} dt \qquad (5)$$

to the relations (1) and (2), and taking the initial conditions to zero, we obtain the following relations

$$P_S\left(\omega\right) \overline{S}_{ij} = Q_S\left(\omega\right) \overline{e}_{ij} \quad P_v\left(\omega\right) \overline{\sigma}_{kk} = Q_v\left(\omega\right) \overline{\varepsilon}_{kk} \qquad (6)$$

here P_S, Q_S, P_v and Q_v are polynomials dependent on ω.

We introduce the complex models of the change of the form and volume as follows:

$$Y_S(\omega) = \frac{Q_S(\omega)}{P_S(\omega)}; \quad Y_S(\omega) = \frac{Q_v(\omega)}{P_v(\omega)}; \tag{7}$$

In this case, the relations (6) take the following form:

$$\overline{S}_{ij} = Y_S(\omega)\,\overline{e}_{ij}; \quad \overline{\sigma}_{kk} = Y_v(\omega)\,\overline{e\varepsilon}_{kk} \tag{8}$$

From (8) we obtain:

$$\overline{\varepsilon}_{ij} = e_{ij} + \frac{1}{3}\overline{e}_{kk}\delta_{ij} = \frac{1}{Y_S}S_{ij} + \frac{1}{3Y_v}\overline{\sigma}_{kk}\delta_{ij} = \frac{1}{Y_S}\sigma_{ij} - \frac{1}{3}\left(\frac{1}{Y_S} - \frac{1}{Y_v}\right)\overline{\sigma}_{kk}\delta_{ij} \tag{9}$$

In a particular case, from (9) we can write the following expressions

$$\begin{array}{l} \overline{\varepsilon}_{11} = \left(\frac{2}{Y_S} + \frac{1}{Y_v}\right)\overline{\sigma}_{11} - \frac{1}{3}\left(\frac{1}{Y_S} - \frac{1}{Y_v}\right)(\overline{\sigma}_{22} + \overline{\sigma}_{33}); \\ \overline{\varepsilon}_{12} = \frac{1}{Y_S}\overline{\sigma}_{12} \end{array} \tag{10}$$

The dependences for elastic material are as follows:

$$\overline{\varepsilon}_{11} = \frac{1}{E}\overline{\sigma}_{11} - \frac{v}{E}(\overline{\sigma}_{22} + \overline{\sigma}_{33}); \;\; \overline{\varepsilon}_{12} = \frac{1}{2\mu}\overline{\sigma}_{12}; \;\; \overline{\theta} = \overline{\varepsilon}_{kk} = \frac{\overline{\sigma}}{K} = \frac{\overline{\sigma}_{kk}}{3K} \tag{11}$$

Theorem 1. *Text of theorem*

$$\lim_{s\to\infty} F(s) = 0. \tag{12}$$

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Torsion of rectangular plate taking into account different modularity

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We consider a rectangular plate loaded along the contour by the distributed twisting pairs taking into account different modularity.

The torque is considered to be positive if it tends to rotate around the outer normal in a clockwise direction.

Our goal is to make up a differential equation connecting the torque with the plate deflection. To do this, from the plate we cut out an element with the base $dxdy$ and height h.We replace the discarded part with torques Hdy and H_1dx. These torques are actually carried out by tangent stresses. We draw up the equivalence conditions as in [2]. To do this we select a strip of width dz on the right edge of the element at a distance z from the median plane. Composing a torque for all forces acting on the element edge, we equate it to the torque Hdy

$$\int_{-h/2}^{h/2} z\tau_{yx}dz = H \tag{1}$$

The equivalence condition for the front edge will be

$$\int_{-h/2}^{h/2} z\tau_{yx}dz = -H_1 \tag{2}$$

Comparing the expressions (1) and (2), we get

$$H = -H_1$$

We obtain the so-called torque reciprocity law for the plate which is a consequence of the law of reciprocity of tangential stresses.

Tangential stresses are expressed through the shear angle as in [1]

In tension

$$\tau_{yx}^{+} = \frac{E^{+}}{2(1+v^{+})}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \tag{3}$$

in compression

$$\tau_{yx}^{-} = \frac{E^{-}}{2(1+v^{-})}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \tag{4}$$

$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = -2z\frac{\partial^2 w}{\partial x \partial y} \tag{5}$$

Taking into account (5) in (3) and (4), we have

$$\tau_{yx}^{+} = -\frac{E^{+}}{1+v^{+}}\frac{\partial^2 w}{\partial x \partial y}z, \quad \tau_{yx}^{-} = -\frac{E^{-}}{1+v^{-}}\frac{\partial^2 w}{\partial x \partial y}z \tag{6}$$

Taking into account the value of τ_{yx}^{+} and τ_{yx}^{-} in (1), we obtain

$$\int_{-h/2}^{0} \tau_{yx}^{+} z dz + \int_{0}^{h/2} \tau_{yx}^{-} z dz = H$$

$$\left[\frac{E^{+}}{(1+v^{+})}\frac{h^3}{24} + \frac{E^{-}}{(1+v^{-})}\frac{h^3}{24}\right]\frac{\partial^2 w}{\partial x \partial y} = -H$$

$$\left[\frac{E^{+}(1-v^{+})h^3}{12(1-v^{2+})} + \frac{E^{+}(1-v^{+})h^3}{12(1-v^{2-})}\right]\frac{\partial^2 w}{\partial x \partial y} = -2H$$

$$\left[D^{+}(1-v^{+}) + D^{-}(1-v^{-})\right]\frac{\partial^2 w}{\partial x \partial y} = -2H \tag{7}$$

where

$$D^{+} = \frac{E^{+}h^3}{12(1-v^{2+})}, \quad D^{-} = \frac{E^{-}h^3}{12(1-v^{2-})}$$

We find out what shape the plate will take during pure torsion. Let a rectangular plate be loaded at all four edges with torgues of constant intensity $H = const$.

The solution of the equation (7) may be taken in the form

$$w = Cxy$$

Having substituted in equation (7), we find the value of the constant

$$C = -\frac{2H}{D^{+}(1-v^{+}) + D^{-}(1-v^{-})}$$

The final expression for the curved surface

$$w = -\frac{2H}{D^{+}(1-v^{+}) + D^{-}(1-v^{-})}xy$$

The equation is a hyperbolic paraboloid. Consequently, in pure torsion, a rectangular plate is bent along the surface of a hyperbolic paraboloid.

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Viscoelastic stresses in a composite pipe

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Let's put a second pipe made of the same material, heated to a temperature T, on a pipe made of a viscoelastic material [1] in a cold state with an internal radius a and an external radius c. The second pipe in the heated state has radii - internal c, external b. After time t, the temperature T cools down. In this case, the second pipe is compressed by the first; self-stresses arise in the composite pipe. This technology is used in practice to increase the strength of pipes. Let's use a cylindrical coordinate system (r, φ, z). A problem of determining the stress-strain state of the considered composite pipe will be a problem of determining natural stresses σ_r, σ_φ, σ_z , strains ε_r, ε_φ and radial displacement u in an entire pipe of radii a and b when there are initial strains in the domain bounded with the radii c and b [2]:

$$\varepsilon_r^* = \varepsilon_\varphi^* = \varepsilon_z^* = \varepsilon^* = \begin{cases} 0, & a \le r < c \\ -\alpha T(t), & c \le r \le b. \end{cases} \tag{1}$$

Here $\alpha-$ is the linear thermal expansion coefficient.

The mechanical properties of pipes are described by the relations of the isotropic linear theory of viscoelasticity [1]. In relation to our problem we have:

$$\sigma_r - \sigma = \int_0^t \Pi\left(t-\tau\right)\, d\left(\varepsilon_r - \varepsilon\right), \qquad \sigma_\varphi - \sigma = \int_0^t \Pi\left(t-\tau\right)\, d\left(\varepsilon_\varphi - \varepsilon\right) \tag{2}$$

or

$$\varepsilon_r - \varepsilon = \int_0^t I\,\left(t-\tau\right)\, d\left(\sigma_r - \sigma\right), \qquad \varepsilon_\varphi - \varepsilon = \int_0^t I\,\left(t-\tau\right)\, d\left(\sigma_\varphi - \sigma\right) \tag{3}$$

We add the following relation to equations (2) (or (3)):

$$\sigma = 3K\left(\varepsilon - \varepsilon^*\right) \ \text{, or } \varepsilon = \varepsilon^* + \frac{\sigma}{3K}\,. \tag{4}$$

Here $\Pi\ (t)$ and $I\ (t)$ are the material functions of relaxation and creeping, which correspond to shear conditions; σ and ε average stress and strain, respectively; K - volumetric deformation modulus. The material of the pipe does not have volumetric rheological properties.

The general formulation of the mathematical problem except equations (2) (or (3)), (4), also contains an equilibrium equation with boundary conditions and Cauchy relations:

$$\frac{\partial \sigma_r}{\partial r} + \frac{\sigma_r - \sigma_\varphi}{r} = 0\,; \quad \left.\sigma_r\right|_{r=a} = 0\,; \quad \left.\sigma_r\right|_{r=b} = 0\,, \tag{5}$$

$$\varepsilon_r = \frac{\partial u}{\partial r}\,, \quad \varepsilon_\varphi = \frac{u}{r}\ . \tag{6}$$

To solve the problems (1),(2), (4)-(6) the approximation method of A.A. Ilyushin [1] is used, which involves the use of a solution to the corresponding elastic problem and the Laplace-Carson integral transform. After some conversions, the solutions of the problem (1),(2),(4)-(6) for movement $u\ (r,\, t)$ is obtained in the form:

$$u\ (r,\, t) = -\frac{a^2 - 3r^2}{r}\,\frac{b^2 - c^2}{b^2 - a^2}\alpha T\left(t\right) + \frac{3}{r}\int_0^t g_2\left(t-\tau\right)\, d\left[\int_a^r \varepsilon^*\left(\tau\right) r dr\right] +$$

$$+3r\frac{b^2 - c^2}{b^2 - a^2}\int_0^t g_{\frac{1}{2}}\left(t-\tau\right)\, d\left(\alpha T\left(\tau\right)\right)\,. \tag{7}$$

Solutions of the problem (1),(2),(4),(6) for stresses σ_r , σ_φ, σ_z , are also obtained.

$$\sigma_r = \begin{cases} \frac{b^2-c^2}{b^2-a^2}\left(1-\frac{a^2}{c^2}\right) M(t), & a \le r < c, \\ \frac{a^2-c^2}{b^2-a^2}\left(1-\frac{b^2}{r^2}\right) M(t), & c < r \le b. \end{cases}$$

$$\sigma_\varphi = \begin{cases} \frac{b^2-c^2}{b^2-a^2}\left(1+\frac{a^2}{r^2}\right) M(t), & a \le r < c, \\ \frac{a^2-c^2}{b^2-a^2}\left(1+\frac{b^2}{r^2}\right) M(t), & c < r \le b. \end{cases}$$

$$\sigma_z = \begin{cases} \frac{2\left(b^2-c^2\right)}{b^2-a^2} N(t), & a \le r < c, \\ \frac{2\left(a^2-c^2\right)}{b^2-a^2} N(t), & c < r \le b. \end{cases}$$

where

$$M(t) = \int_0^t \Pi(t-\tau)\, d\left(\alpha T(\tau)\right), \quad N(t) = M(t) + \frac{3}{2}\int_0^t g_{\frac{1}{2}}(t-\tau)\, dM(\tau).$$

When writing the solution of the viscoelastic problem, the solution of the elastic problem given in [2], was used. The functions $g_2(t)$ and $g_{1/2}(t)$, included in (7), were introduced by A.A. Ilyushin [1]. A.A. Ilyushin formulated a system of experiments to determine them. However, analytical formulas obtained for these functions, are presented in [3].

Let's note the following. The strain components ε_φ and ε_r can be determined using (6) and (7). At the same time, ε_φ and ε_r are also determined by formulas (3) taking into account (4). Comparison of the obtained expressions for ε_φ and ε_r by these two approaches, made it possible to derive relations connecting the functions $g_2(t)$ and $g_{1/2}(t)$.

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Study of mass exchanges in gas formation in porous media

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The cases of artificial gas generation in layers during displacement processes in porous media are more promising and have certain effects [2,4]. The gas formed here creates displacements in thermobaric conditions in porous media. A deeper study of these processes is one of the important issues.

In this work, the kinetics of gas formation in the mutual contact of the composite systems intended for the displacement process in the porous medium and the special cases of the migration process within the medium are investigated.

Mathematical formulation of the problem

As it is known, the pressure dynamics during gas separation in solutions where gas is formed in mutual contact increases monotonically at different rates depending on the characteristics of the systems [1,3].

In the presence of porous media and fluids, the pressure dynamics is monotonous, accompanied by a decrease in pressure at the final stage.

In this case, the pressure change in the formation of gas bubbles is written in the form of kinetic equations as follows:

$$\frac{dP}{dt} = a_1 - a_2 P(t) - a_3 P^2(t-\tau)$$

a_2 and a_3 – coefficients that take into account the gas dissolved in the liquid and already formed and diffused gas molecules; τ is the characteristic diffusion time.

Let's look at the solution of the equation under the following conditions

$$P(t) = \varphi_0(t), \; t_0 - \tau \le t \le t_0$$
$$\varphi_0(t) = \frac{1}{2a\sqrt{\pi\, t}} e^{-\frac{x^2}{4a^2 t}}$$

$$\begin{cases} \frac{dP}{dt} = a_1 - a_2 P(t) - a_3 \varphi_0^2(t-\tau) \\ t_0 \le t \le t_0 + \tau, \; \rho(t_0) = \varphi_0(t_0); \qquad [t_0, \; t_0 + \tau] \end{cases}$$

The solution is sought in the form of the following function:

$$P_1(t) = e^{-a_2(t-t_0)} \varphi_0(t_0) + \int_{t_0}^{t} e^{-a_2(t-s)} \left(a_1 - a_3 \varphi_0^2(s-\tau)\right) ds$$

$$t_0 + \tau \le t_0 \le 2\tau, \qquad P(t_0 + \tau) = P_1(t_0 + \tau)$$

$$\frac{\partial P}{\partial t} = a^2 \frac{\partial^2 P}{\partial x^2}, \qquad t > 0, \qquad P = \varphi_0(t)$$

Here $\varphi_0(t,x) = \frac{1}{2a\sqrt{\pi\, t}} e^{-\frac{x^2}{4a^2 t}}$

According to the Taylor separation, the solution to the first approximation can be written as follows:

$$P_1(t_1^\tau) = \frac{a_1}{a_2}\left(1 - e^{-a_2(t-t_0)}\right) + \frac{1}{2a\sqrt{\pi t_0}} e^{-a_2(t-t_0) - \frac{x^2}{4a^2 t_0}} -$$

$$-\frac{a_3}{4\pi}\left\{\left(1 - \frac{a_2 x^2}{2a^2}\right)\ln\frac{t-\tau}{t_0-\tau} + \left(a_2 - \frac{a_2^2 x^2}{4a^2}\right)\cdot(t-t_0) + \frac{x^2}{2a^2}\frac{(t-t_0)}{(t-\tau)\,(t_0-\tau)} + ...\right\}$$

Here

$a_1 = kv_1$ rate of pressure change; v_1- bubble formation velocity;

$a_2 = \alpha\, v_2$ that takes into account the relative solubility of the gas;

$a_3 = v_3$ coefficient, which takes into account the influence of excessively generated gas on the formation of new bubbles.

In the second stage, the diffusion of gas bubbles into the environment was considered. Here, the coefficient that takes into account the diffusion can be found from the experimental results.

Considering the average velocity of gas particles in the medium and the velocity of the diffusing particles, we can write the length of the free movement path with the known following expression.

$$\lambda = \frac{1}{\sqrt{2}\sigma n}$$

If we direct only the z-axis orthogonally to the area S in the pore in the medium, then the complete motion of the particles can be written as follows:

$$\phi(z) = \frac{dN}{dt} = -\frac{1}{3}\langle v\rangle \lambda \frac{\partial n(z)}{\partial z} S$$

Since this structurally coincides with Fick's first law, the diffusion coefficient is defined as follows:

$$D = \frac{1}{3}\langle v\rangle \lambda$$

Considering this, we can write the diffusion as follows:

$$\langle v\rangle \sim \sqrt{\frac{T}{\mu}}, \lambda \sim \frac{1}{\sigma n} \sim \frac{T}{\sigma P}, D \sim \frac{1}{\sigma n}\sqrt{\frac{T}{\mu}} \sim \frac{T^{\frac{3}{2}}}{\sigma P\sqrt{\mu}}$$

As a result of the conducted studies, it was determined that the pressure increase in the reacting media and its value is settled and that the pressure changes from the stoichiometric reaction in the solutions occur at different rates depending on the type of water.

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THEORY OF PROBABILITY AND MATHEMATICAL STATISTICS

On convergence of age-dependent branching process with emigration

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We define multi-type age-dependent branching process with emigration by analogy with [1]

As in the multi-type process without emigration we have n types of particles $T_1, T_2, ..., T_n$, Each T_i-th type particle has random duration of, presence in process τ_i with distribution function

$$P(\tau_i \leq t) = G^i(t),\ G^i(0+) = 0.$$

We will assume that $G^i(t)$ is absolutely continuous. At the end of its presence particle of any type is transformed into arbitrary number of particles of any types or emigrate. And let us define type T_0 as a type of particles which emigrated . Conditional probability (if transformation took place when the age attained by the original particle was u) $p^i_\alpha(u)$ of transformation into a set consisting of α_i T_i-th type particles, $i = \overline{0, n}$ where $\alpha = (\alpha_0, \alpha_1, \alpha_2, ..., \alpha_n)$ is $n + 1$ –dimensional vector and α_0 can be 0 (in the case other components of vector take non-negative integer values) or 1 (in this case other components of vector are zeros). The evolution of the particle is defined by the joint distribution of random variable τ_i and random vector $v_i = (v_i^0, v_i^1, ..., v_i^n)$ which characterize the progeny of this particle.

A particle of type T_0 at any moment transforms into itself, so it can be viewed as T_0-th type particle has degenerate at zero life duration distribution function ($G^0(t) = 1,\ \forall t > 0$).

Also we will assume that $p_1^0(u) = 1$, $p_1^0(u) = 0$, for all $u \geq 0$ and all vector $\alpha \neq (1, 0, ..., 0)$. Vector $\mu_i(t) = (\mu_i^0(t), \mu_i^1(t), ..., \mu_i^n(t))$ denotes the number of particles of types $T_0, T_1, ..., T_n$, at the moment t, under the condition, that at the initial moment, there existed one T_i th type particle.

And let's also assume, that vector $\mu^i(t)$ is right continuous. Let's denote by $P^i(*)$ conditional probability under the condition that at the initial moment of time there existed one particle of type T_i.

We introduce generating functions

$$h^i(t,s) = \sum_\alpha P_\alpha^i(t) s^\alpha \quad \text{and} \quad F^i(t,s) = \sum_\alpha P^i(\mu(t) = \alpha) s^\alpha, \quad i = \overline{1,n},$$

$$s = (s_0, s_1, ..., s_n), \quad s^\alpha = (s_0^{\alpha_0}, s_1^{\alpha_1}, ..., s_n^{\alpha_n}),$$

$$F(t,s) = (F^0(t,s), F^1(t,s), F^n(t,s)), \quad h(t,s) = (h^0(t,s), h^1(t,s), h^n(t,s))$$

It's clear that $F^0(t,s) = h^0(t,s) = s_0$ for all t.

In this work under certain conditions on generating functions $h(t,s)$ and $F(t,s)$ age-dependent branching processes with emigration as a part of the same process are investigated.

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On solvability of some quasilinear elliptic systems of higher order

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Let Ω be a bounded domain located in N dimensional Euclidean space R^N of the points $x = (x_1, ..., x_N)$ with the boundary of the class $c^{2m}, u = u(x), Du = Du(x)$ be a gradient of $\partial\Omega$ the function $u(x)$, $\overrightarrow{u}(x) = (u_1(x), ..., u_n(x))$, $D\overrightarrow{u}(x) = (Du_1(x), ..., Du_n(x))$ .We will write that $\overrightarrow{u} \in \overrightarrow{L}_p(\Omega)$ if $u_j \in L_p(\Omega)$, $j = 1, .., n$ and $\|\overrightarrow{u}\|_{p;\Omega} = \sum\limits_{j=1}^{n} \|u_j\|_{Lp(\Omega)}$.

We are given the operator

$$\overrightarrow{L}\overrightarrow{u} = \left\{ \sum_{|\alpha|=2m} \sum_{j=1}^{n} a^k_{\alpha j}(x, \overrightarrow{u}, D\overrightarrow{u}, ..., D^k\overrightarrow{u}) D^\alpha u_j, \quad v = \overline{1,n} \right\}.$$

Let us consider the boundary value problem

$$\begin{cases} \overrightarrow{L}\overrightarrow{u} = \overrightarrow{f}(x, \overrightarrow{u}, D\overrightarrow{u}, ..., D^{2m-1}\overrightarrow{u}), & x \in \Omega, \\ \overrightarrow{B}_i\overrightarrow{u}\Big|_{\partial\Omega} = \overrightarrow{\varphi}(x), i = 0, 1, ..., m-1, & x \in \partial\Omega \end{cases} \tag{1}$$

This problem is studied in the class of real functions from the Sobolev space $\overrightarrow{w}^2_p(\Omega)$ with $p(2m + n) > 2n$ provided that there exists a priori estimation $\|\overrightarrow{u}\|_{\overrightarrow{w}^2_p(\Omega)}$ in the space $\overrightarrow{w}^2_p(\Omega)$ and $m > \frac{n}{2}$.

Assume that the following conditions are fulfilled (see [1]).

A1) the components of the vector $\overrightarrow{f}$ of the function $f_\nu(x, \overrightarrow{\xi}_0, \overrightarrow{\xi}_1, ..., \overrightarrow{\xi}_{2m-1})$, $\nu = \overline{1,n}$ are determined on $\overline{\Omega} \times R^n \times R^{nN_1} \times ... \times R^{nN_{2m-1}}$ and is Caratheodorian

A2) Let $m > \frac{n}{2}$. Denote by l_0 the least entire positive number greater or equal to $m - \frac{n}{2}$ and let $\overrightarrow{\xi}_* = \left\{ \overrightarrow{\xi}_\gamma | \, |\gamma| < m - \frac{n}{2} \right\}$,

$$\sum_{\nu=1}^{n} \left| f_\nu(x, \overrightarrow{\xi}_0, ..., \overrightarrow{\xi}_{2m-1}) \right| \leq b(x, \overrightarrow{\xi}_*) + \sum_{l=l_0}^{2m-1} b_l(x, \overrightarrow{\xi}_*) \left| \overrightarrow{\xi}_l \right| \mu_l$$

almost for all , $\overrightarrow{\xi}_0 \in R^n, ..., , \overrightarrow{\xi}_{2m-1} \in R^{nN_{2m-1}}$ with such non-negative Caratheodory functions b, b_l that for any $r \geq 0$

$$b_r(x) \equiv \sup\left\{ b(x, \overrightarrow{\xi}_*) \left| \left|\overrightarrow{\xi}_*\right| = \sum_{|\gamma|<m-\frac{n}{2}} \left|\overrightarrow{\xi}_\gamma\right| \le r \right.\right\}$$

belongs to $L_p(\Omega)$ with $p > 1$ and $p(2m+n) > 2n$ the functions $b_{l,r}(x) \equiv \sup\left\{ b_l(x, \overrightarrow{\xi}_*) \left| \left|\overrightarrow{\xi}_*\right| \le r \right.\right\}$ belong to $L_{q_l}(\Omega)$ with $q_l > p$ for $l = l_0, ..., 2m-1$.

A3) Let

$$0 \le \mu_l < \mu_l^* = \frac{2m+n}{n+2(l-m)} - \frac{1}{n+2(l-m)}\frac{2n}{q_l}$$

for $l = l_0, ..., 2m-1$, if $m > \frac{n}{2}$.

A4) Let $m > \frac{n}{2}$ be such an integer $k \ge 0$ that $m-k > \frac{n}{2}.\overrightarrow{L}$ be a quasilinear elliptic operator of order $2m \ge 2$ with some $k, 0 \le k \le 2m-1$, the coefficient $a_{\alpha j}^{\nu}$ be real and continuous functions on $\overrightarrow{\Omega} \times R^n \times ... \times R^{nN_k}$ and $\overrightarrow{B}_i$ $(i = 0, 1, ..., m-1)$ be linear boundary differential operators of orders $m_i \le 2m-1$, respectively, with real coefficients.

Let the operator $\overrightarrow{L_v}\overrightarrow{u}$ linear with respect to $\overrightarrow{u}(x)$ equal to

$$\overrightarrow{L_v}\overrightarrow{u} = \left\{ \sum_{|\alpha|=2m}\sum_{j=1}^{n} a_{\alpha j}^{\nu}(x, \overrightarrow{v}, ..., D^k\overrightarrow{v}) D^\alpha u_j, \quad v = \overline{1,n} \right\}$$

for any function $\overrightarrow{v} \in \overrightarrow{c}^k(\overline{\Omega})$ be such a linear operator that the linear boundary value problem

$$\left\{ \begin{array}{c} \overrightarrow{L_v}\overrightarrow{u} = \overrightarrow{g}(x), \quad x \in \Omega, \\ \overrightarrow{B}_i \overrightarrow{u}\Big|_{\partial\Omega} = \overrightarrow{\varphi}(x), (i = 0, 1, ..., m-1), \quad x \in \partial\Omega \end{array} \right. \tag{2}$$

be coercive in the space $\overrightarrow{W}_p^2(\Omega)$ i.e. the following a priori estimation:

$$\|\overrightarrow{u}\|_{\overrightarrow{w}_p^2(\Omega)} \le c\left(\|\overrightarrow{g}\|_{p;\Omega} \sum_{i=0}^{m-1} \|\overrightarrow{\varphi}_i\|_{\overrightarrow{W}_p^{2m-m_{i-1/p}}(\partial\Omega)} + \|\overrightarrow{u}\|_{p;\Omega} \right)$$

with a positive constant C independent of $\overrightarrow{g} \in \overrightarrow{L}_p(\Omega)$ of $\overrightarrow{\varphi}_i \in \overrightarrow{W}_p^{2m-m_{i-1/p}}(\partial\Omega)$ and of the solution $\overrightarrow{u} \in \overrightarrow{L}_p(\Omega)$ of linear problem (2), be fulfilled.

In the paper, we cite theorems on a priori solutions of solutions $\|\overrightarrow{u}\|_{\overrightarrow{W}_p^2(\Omega)}$ in the norm of the Sobolev space $\overrightarrow{W}_p^2(\Omega)$ expressed by the norm $\|\overrightarrow{u}\|_{\overrightarrow{W}_p^2(\Omega)}$.

The proof of solvability of boundary value problems is carried out on the basis of the cited theorems on a priori estimations using the Leray-Schauder method.

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On the law of large numbers of the moments of the first intersection of the level by a random walk described by an autoregressive process with a random coefficient

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Let $\xi_n,\ n \geq 1$ be a sequence of independent identically distributed random variables determined on the probability space $(\Omega, \mathcal{F}, P)$.

Let us consider the first order autoregression process $(AR\,(1))$ in the following form

$$X_n - m = \beta\,(X_{n-1} - m) + \xi_n, \quad n \geq 1$$

where the initial value X_0 of the process is independent of the innovation $\{\xi_n\}$, and $m,\ \beta \in R = (-\infty, \infty)$ some fixed numbers.

At present great attention is paid to the study of limit theorems for Markov random walks described by first-order autoregression process (RCAR (1)) with the random coefficient β.

The RCAR (1) processes were first introduced and studied in the work [1].

Random coefficient autoregression processes have applications in time series theory.

In the paper, in particular, we prove the law of large numbers for the family of the moments of the first intersection of a level by the sums of values of a random coefficient autoregression process.

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On the law of large numbers sums of the values of first-order autoregressive process with random coefficient $(RCAR(1))$

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Let $\xi_n, n \geq 1$ be a sequence of independent identically distributed random variables determined on the probability space (Ω, F, P).

As is known ([1],[2]) the random coefficient first order auto regression process

$((RCAR(1))$ is the solution of a recurrent equation of the form

$$X_n = \beta X_{n-1} + \alpha \xi_n, \quad n = 1, 2... \tag{1}$$

where $\beta = \beta(\omega)$ and $\alpha = \alpha(\omega) \quad \omega \in \Omega$, are some random variables.

We will assume that the initial value X_0 of the process is independent of the innovation $\{\xi_n\}$, and random variables β, α are independent between them selves and do not depend on X_0 and ξ_n for all $n \geq 1$.

Model (1) of ($RCAR(1)$) process arises in theoretical and also in practical problems of the theory of time series ([1],[2]).

Note that in the case when α and β are fixed real numbers, the model of the form (1) coincides with the random coefficient first order auto, regression process of the form

$$Y_k = \theta Y_{k-1} + \eta_n, \tag{2}$$

where θ is a fixed number, and $\eta_n, n \geq 1$ are independent identically distributed random variables ([3]).

Recently the theory of nonlinear renewal has been intensively developed for Markov random walks described by the nonrandom coefficient first-order auto-regression process and also with random coefficients [3,4].

Let us consider the following Markov random walk described by the ($RCAR(1)$) process of the form (1)

$$S_n = \sum_{k=0}^{n} X_k, \quad n \geq 1 \tag{3}$$

It is clear that in the case of the scheme (2) ($X_k = Y_k$) for $\theta = 0$ the sum of the form S_n (3) forms a usual classic random walk described by the sums of independent identically distributed random variables.

Theorem 1. *Let $E\,|X_0| < \infty$, $E\,|\xi_1| < \infty$ and $P(0 \leq \beta < 1-\varepsilon) = 1$ for some $\varepsilon \in (0,1)$ and $P(\alpha \neq 0) = 1$.*

Then, 1) in probability $\frac{1-\beta}{\alpha}\frac{S_n}{n} \xrightarrow{P} = m = E\xi_1$ as $n \to \infty$.

2) If $\frac{X_n}{n} \xrightarrow{a \cdot s} 0$, then $\frac{1-\beta}{\alpha}\frac{S_n}{n} \xrightarrow{a \cdot s} m$, $n \to \infty$.

Theorem 2. *Let $\sigma^2 = D\xi_1 < \infty$ and the conditions of theorem 1 be fulfilled Then*

$$\lim_{n\to\infty} P\left(\frac{\sqrt{n}}{\sigma}\left(\frac{1-\beta}{\alpha}\frac{S_n}{n} - m\right) \leq x\right) = \Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-y^2/2}dy, \ x \in R.$$

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Generalized entropy optimization distributions: a case study application

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This article investigates the construction of Generalized Entropy Optimization Methods (GEOM‘s). These methods are developed in over really investigations which allow to obtain in particularly distributions in the forms of MaxMaxEnt, MinMaxEnt, MaxMinxEnt, and MinMinxEnt distributions so-called as Generalized Entropy Optimization distributions (GEOD‘s).

Construction of Generalized Entropy Optimization Distributions (GEOD) is communicated with requirements of probability theory, information theory, Stochastic Differential Equation (SDE) and other scientific areas. This article mainly describes about mentioned distributions, which are MaxMaxEnt, MinMaxEnt, MinMinxEnt and MaxMinxEnt distributions. MaxMaxEnt distribution constructed by virtue of MaxEnt distribution in the following form: First, MaxEnt distribution should be constructed in order to obtain MaxMaxEnt distribution. In this case, using Shannon entropy measure maximizes by using linear independent conditions generated by given frequencies of statistical data and linear independent characterizing moment functions and moment values. The number of linear independent conditions must be smaller than the number of given frequencies of data. Maximizing problem solved by Lagrange multipliers method leads for obtaining distribution which is MaxEnt distribution.

By using linear conditions generated n number of frequencies of statistical data and m number of linear independent characterizing moment functions MaxEnt distribution is obtained. Number of Maxent distributions is $\sum_{r=1}^{m}\binom{r}{m}$, where $m+1<n$ and every combination of m things r at a time defines $\binom{r}{m}$ number of MaxEnt distributions.

Suppose the set of N MaxEnt distributions is described as M. Then GEODs can be obtained as following:

MaxMaxEnt distribution is MaxEnt distribution from M Shannon entropy value of which is the greatest among distributions from M.

MinMaxEnt distribution is MaxEnt distribution from M Shannon entropy value of which is the least among MaxEnt distributions of M.

Acquiring statistical data allow to obtain approximated distribution and density function of random variable by virtue of GEOD's.

It is known that mentioned density function is solution of Kolmogorov equation. Consequently, by using approximate solution of SDE it is possible to obtain approximate solution of Kolmogorov equation.

Analogically as obtaining of MaxEnt distributions by using r linear conditions generated n number of characterizing moment functions it is possible to obtain MinxEnt distribution by minimizing Kullback-Leibler entropy measure subject to $r+1$ conditions by using Lagrange multipliers method.

Let us set of N number MinxEnt distribution as M_x. Then MaxMinxEnt distributions is MinxEnt distribution from M_x Kullback-Leibler measure value of which is the greatest among distributions MinxEnt distribution from M_x. MinMinxEnt distribution from M_x is MinxEnt distribution from M_x Kullback-Leibler measure of which is the least value among distributions of M_x.

It is used some real dataset in order to show the existence of construction GEOD's. Aransas-Wood Buffalo population of whooping cranes. The population data for the whooping cranes. These whooping cranes nest in Wood Buffalo National Park in Canada and winter in Aransas National Wildlife Refuge in Texas. In order to show the construction of GEOD's the dataset was used.

Conclusion. This article aimed to show the existence of constructing Generalized entropy optimization methods. In assessment of unknown distribution functions, it showed good performance with different combinations of characterizing moment functions. To obtain GEOD based on MaxMaxEnt and MinMaxEnt we used some real dataset from literature and approached the selected distribution in order to find best approximation to provided dataset.

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